GRAPH THEORY NOTES OF NEW YORK

LV

Dedicated to the memory of

Michael L. Gargano

New York Academy of Sciences

Editors:
John W. Kennedy
Louis V. Quintas

(2008)

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Graph Theory Notes of New York publishes short contributions and research articles in graph theory, its related fields, and its applications.

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This issue includes papers presented at

Graph Theory Day 55

held at

The Department of Mathematics

Hartwick College

Oneonta, New York

Saturday, May 10, 2008

CONTENTS

INTRODUCTORY REMARKS

This issue of *Graph Theory Notes of New York* is dedicated to Michael L. Gargano (1947–2008). Mike passed away on May 31, 2008 at the peak of his creative and inspirational activity. As a member of our editorial board, Mike was an active supporter of *Graph Theory Notes of New York* and of the associated Graph Theory Day meetings. He held a joint appointment in the Computer Science and Mathematics Departments at Pace University, where he had been for 30 years. For those who knew Mike, his ability to find a wide range of interesting mathematical topics to study was impressive. Mike's interests were indeed diverse. He explored many applications of mathematics (graph theory and combinatorics) and computer science to archaeology, anthropology, biology, history, literature, psychology, political science, sociology, genetics, and the use of discrete mathematics in the combined areas of terrorism and security. One of Mike's favorite tools was to apply genetic algorithms to diverse areas. Simply stated Mike was a scholar.

This facet of Mike Gargano was indeed an inspiration to his colleagues and to his students. Another facet was his role model as a person. He was generous to all that came into contact with him, almost always in a good mood, and as many can relate stories about, he was a teller and appreciator of jokes. He was a person one enjoyed being with. The shock of his passing is over, the mourning is subdued but is still deep, and it is now time to realize how fortunate we are to have had the joy of knowing Mike.

Graph Theory Day 55 was hosted by Hartwick College, Oneonta, New York. The local organizer was Gary E. Stevens of Hartwick College. The invited speakers were Ralph Grimaldi and Stephen Hedetniemi. Regretably, because of a last minute family emergency, Steve Hedetniemi was unable to attend. Instead, Steve made available the content and materials for his talk and we appreciated its delivery by Gary Stevens.

Potential contributors of articles to *Graph Theory Notes of New York* are asked to consult the inside of the back cover for submission instructions. Those that enjoy Graph Theory Days are encouraged to ask their institutions to host one of these biannual events so that this twenty-eight year tradition continues to flourish.

As always we thank all the supporters of *Graph Theory Notes of New York* and Graph Theory Days.

JWK/LVQ
New York
November 2008

GRAPH THEORY DAY 55

Organizing Committee

Gary E. Stevens (Hartwick College)
John W. Kennedy (Queens College, CUNY)
Michael L. Gargano and Louis V. Quintas (Pace University)

Graph Theory Day 55, sponsored by the Mathematics Section of the New York Academy of Sciences, was organized and hosted by the Department of Mathematics, Hartwick College, Oneonta, New York on Saturday, May 10, 2008.

The featured presentations at Graph Theory Day 55 were:

The Fibonacci, Pell, and Jacobsthal Numbers within Graph Theory
Ralph P. Grimaldi
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, Indiana, U.S.A.

Backup Domination in Graphs
Stephen T. Hedetniemi
Department of Mathematics
Lafayette College
Easton, Pennsylvania, U.S.A.
[due to an emergency, this talk was delivered by Gary E. Stevens of Hartwick College]

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[GTN LV:1]

VARIABLE NEIGHBORHOOD SEARCH FOR EXTREMAL GRAPHS: 25. PRODUCTS OF CONNECTIVITY AND DISTANCE MEASURES

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Abstract

Upper bounds for products of four measures of distance in graphs: diameter, radius, average eccentricity, and remoteness with three measures of connectivity: vertex connectivity, algebraic connectivity, and edge connectivity are analyzed. Twelve conjectures were obtained by using AGX 2 software. Eight of them are proved to be correct, three are disproved, and one remains an open problem.

1. Introduction

Let $G = (V, E)$ be an undirected connected graph of order $n = |V|$ and size $m = |E|$. Let $d(v)$ denote the degree of $v \in V$. The *minimum* and *maximum degrees* of G are denoted by δ and Δ, respectively.

An *edge* (*vertex*) *cut* in G is a set of edges (vertices) of G deletion of which disconnects G or reduces it to a single vertex. The size of an edge (vertex) cut is the number of edges (vertices) it contains. The *edge* (*vertex*) *connectivity* is the size of a smallest edge (vertex) cut. The *Laplacian matrix* L of G is defined by $L = Deg - A$, where Deg is the diagonal matrix of vertex degrees and A is the adjacency matrix of G. The *algebraic connectivity* a is the second smallest eigenvalue of L.

Let $d(u, v)$ denote the *distance* between vertices u and v. The *eccentricity* of a vertex v in G is the maximum distance from v to any other vertex in G. The *diameter* D of G is the maximum eccentricity. The *radius* r of G is the smallest eccentricity. The *average eccentricity* is denoted by *ecc*. The *transmission* of a vertex v in a graph G is the sum of the distances between v and all other vertices of G. The transmission is said to be normalized if it is divided by $n - 1$. The *remoteness* ρ is the maximum normalized transmission, or, in other words, the largest average distance between any vertex and all vertices.

Let K_n, C_n, and P_n denote, respectively, the complete graph, the cycle graph, and the path graph of order n. A set M of disjoint edges is called a *matching*; a matching that covers all vertices is a *perfect matching*.

Distance and connectivity measures in graphs appear to be antinomic: Roughly speaking, the larger the distance the smaller the connectivity, and conversely. This suggests the interest in studying products of distance and connectivity measures.

In the thesis **[1]** (see also **[2]** for a summary) a systematic comparison was made on relations between pairs of graph invariants selected from a set of 20 invariants that included the connectivity measures, ν, a, and κ, as well as the distance measures D, r, *ecc,* and ρ. The general form of these relations, called AGX Form 1, is

$$\underline{b}(n) \le i_1 \oplus i_2 \le \bar{b}(n),$$

where i_1 and i_2 are graph invariants, $\oplus$ denotes one of the four operations $-$, $+$, $/$, and $\times$; $\underline{b}(n)$ and $\bar{b}(n)$ are lower and upper bounding functions for $i_1 \oplus i_2$ that depend on the order n (or number of vertices) of the graphs under consideration. These bounding functions are required to be best possible in the strong sense; that is, for all n (except possibly for very small values due to border effects) there is a graph such that the lower (upper) bound is attained. In this paper, we focus on results from that study that concern the upper bound on a product of a connectivity measure and a distance measure, among those cited above. More precisely, we study relations of the form

(1) $$i_1 \oplus i_2 \le \bar{b}(n),$$

where $i_1 \in \{\nu, a, \kappa\}$ and $i_2 \in \{D, r, ecc, \rho\}$.

The relationships of the form **(1)** that are discussed and proved in this paper were obtained, as well as many others **[1]**–**[3]**, with the system AGX 2 **[4]**. This *discovery system* is described, together with its results, in a series of papers (see **[2]**,**[5]** for references), to which the present paper also belongs. It is based on the following observation: A large variety of problems in extremal graph theory can be viewed as parametric combinatorial optimization problems defined on the family of all graphs (or some restriction thereof) and solved by a generic heuristic. The parameter is usually the order n of the graph considered (sometimes the order n and the size m). The heuristic fits in the *variable neighborhood search* metaheuristic framework **[6]**–**[8]**. Presumably, extremal graphs are found by performing a series of local changes (removal, addition, or rotation of an edge, etc.) until a local optimum is reached, then, by applying increasingly large perturbations, followed by new descents; if a graph better than the incumbent one is found, the search is recentered there. After the parametric family of extremal graphs has been found, relationships between graph invariants may be deduced from them by using various *data mining* techniques **[9]**. These include (1) a numerical method based on *principal component analysis,* which yields a basis of affine relations between the graph invariants considered; (2) a geometric method that uses a gift-wrapping algorithm to find the convex hull of extremal graphs viewed as points in the invariant space; facets of this convex hull give inequality relations; (3) an algebraic method that recognizes families of graphs and then exploits a database of formulæ giving expressions of invariants as functions of n on these families; substitution in **(1)** then leads to linear or nonlinear conjectures. As mentioned above, a systematic comparison of 20 invariants was made in **[1]**. The results summarized in **[2]** are available in detail on the website <http://www.gerad.ca/~agx>, and are currently being proved in a series of papers (see **[5]** for references).

The AGX 2 software produced the following conjectures **[1]** on the upper bounds of products of one of $\{\nu, a, \kappa\}$ with one of $\{D, r, ecc, \rho\}$.

Table 1: Conjectures

	D	r	ecc	ρ
ν	$2n-4$	$4 \cdot \lfloor \frac{n}{2} \rfloor - 4$	$2n-4$ if n is even $2n-5+\frac{2}{n}$ if n is odd	$n-1$
a	$2n-4$	$4 \cdot \lfloor \frac{n}{2} \rfloor - 4$	$2n-4$ if n is even $2n-5+\frac{2}{n}$ if n is odd	$n-1$
κ	$2n-4$	$4 \cdot \lfloor \frac{n}{2} \rfloor - 4$	$2n-4$ if n is even $2n-5+\frac{2}{n}$ if n is odd	$n-1$

We proved eight of these 12 claims and disproved three of them; one conjecture remains an open problem. This is summarized in the following table.

Table 2: Summary

	D	r	ecc	ρ
ν	proved	proved	proved	proved
a	proved	proved	proved	proved
κ	disproved	open	disproved	disproved

Each of the following sections is dedicated to one of the distance measures: diameter, radius, average eccentricity, and remoteness. Each of these invariants is combined with the three connectivity measures.

2. Connectivity and Diameter

In this section, we prove the upper bounds on $\nu \cdot D$ and $a \cdot D$ and give a counterexample for the conjectured upper bound on $\kappa \cdot D$.

> **Theorem 1:** Let G be a connected graph on $n \geq 3$ vertices with vertex connectivity ν and diameter D. Then $\nu \cdot D \leq 2n-4$, with equality if and only if G is K_3 or $K_n \backslash M$, where M is a matching.

Proof: If $n = 3$, then G is K_3 or P_3, and equality holds in both cases.

If $G \cong K_n \backslash M$, where M is a matching, then $\nu = n-2$ and $D = 2$, so equality holds.

If $G \cong K_n$, the claim obviously holds, so suppose that $G \ncong K_n$. Let x and y be two vertices at distance D. Note that the set of vertices, V, can be decomposed into $V = V_0 \cup V_1 \cup \ldots \cup V_D$, where V_i is the set of vertices at distance i from x. Obviously, V_i is a cut-set of vertices of G for each i. Hence, $\nu \leq |V_i|$ for each $i = 1, \ldots, D-1$. Also, $|V_D| \geq |V_0| = 1$. Hence, $2 + (D-1) \cdot \nu$, or equivalently, $D \cdot \nu \leq n-2+\nu$. Since $\nu \leq n-2$ ($G \ncong K_n$), then $D \cdot \nu \leq 2n-4$.

If equality holds, then $\nu = n-2$ and $D = 2$, but then $\delta = n-2$ and $D = 2$, which implies that $G \cong K_n \backslash M$, where M is a matching. ■

To prove the upper bound on $a \cdot D$, we need an intermediate result. This is formulated in the following lemma.

> **Lemma 2:** If $G \cong K_n \backslash M$, where M is a matching, then $a = n-2$.

Proof: It is shown in **[10]** that for a complete k-partite graph $G = (V, E)$ with vertex set partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$, then $a(G) = n - \max_{i=1,\ldots,k}|V_i|$. Now, $K_n \backslash M$ can be seen as a complete $(n-p)$-partite graph for which $\max_{i=1,\ldots,n-p}|V_i| = 2$. Thus the lemma follows. ■

> **Theorem 3:** Let G be a connected graph on $n \geq 3$ vertices, $G \ncong K_3$, with algebraic connectivity a and diameter D. Then $a \cdot D \leq 2n-4$ with equality if and only $G \cong K_n \backslash M$, where M is a matching.

Proof: Using Lemma 2, it is easy to see that equality holds if $G \cong K_n \backslash M$.

If $G \cong K_n$ then **[11]** $a = n$ and $D = 1$. Hence, $a \cdot D = n \leq 2n-4$, with equality if and only if $n = 4$, that is, $G \cong K_4$.

Now suppose that $G \ncong K_n$, then **[11]** $a \leq \nu$. Thus, using Theorem 1, we have $a \cdot D \leq \nu \cdot D \leq 2n-4$ with equality if and only $G \cong K_n \backslash M$. ■

Consider, now, the conjecture for the upper bound on $\kappa \cdot D$.

> **Conjecture 4:** Let G be a connected graph on $n \geq 3$ vertices with edge connectivity κ and diameter D. Then $\kappa \cdot D \leq 2n-4$, with equality if and only if $G \cong K_n \backslash M$, where M is a matching. □

The above conjecture is not true.We give a simple counterexample. Let $G_1, G_2, \ldots, G_l$ be disjoint graphs. By $G = G_1 + G_2 + \ldots + G_l$ we mean the *sequential join*, which is defined as follows:

$$V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_l), \text{ and}$$

$$E(G) = E(G_1) \cup \ldots \cup E(G_l) \cup \{xy : x \in V(G_i), y \in V(G_{i+1}), i = 1, \ldots, l-1\}.$$

The graph $H = K_1 + K_{25} + K_5 + K_5 + K_{25} + K_1$ contains $n = 62$ vertices and satisfies $\kappa(H) = 25$ and $D(H) = 5$. Thus, $\kappa \cdot D = 125 > 120 = 2n-4$ and the graph H is a counterexample to Conjecture 4.

3. Connectivity and Radius

In this section, we prove the upper bounds on $\nu \cdot r$ and $a \cdot r$, and give the conjecture for the upper bound on $\kappa \cdot r$ that remains open. First, we need an intermediate result that we state and prove in the following lemma.

Lemma 5: Let G be a connected graph on an odd number of vertices $n \geq 5$ with vertex connectivity ν, minimum degree $\delta = n-3$, and maximum degree $\Delta \leq n-2$. Then $\nu = n-3$ if and only if the complement $\overline{G}$ of G does not contain C_4.

Proof: Suppose $\nu = n-3$. From $\delta = n-3$ and $\Delta \leq n-2$, it follows that $\overline{G}$ consists of disjoint paths and cycles. Suppose the lemma does not hold, that is, $\overline{G}$ contains a cycle C_4. Let v_1, v_2, v_3, and v_4 denote the vertices of one C_4 from $\overline{G}$. Therefore, in the graph G, among the four vertices, v_1 is adjacent only to v_3, and v_2 is adjacent only to v_4. Hence, the set of vertices $V\backslash\{v_1, v_2, v_3, v_4\}$ is a cut in G. Thus, $\nu \leq n-4$, which is a contradiction.

Suppose now that the complement $\overline{G}$ does not contain any C_4. Let V' be a smallest vertex cut set in G. We want to prove that $|V'| = n-3$. Note that $|V'| \leq n-3$, since $|V'| = \nu \leq \Delta \leq n-3$. Furthermore, $\overline{G}\backslash V'$ is a connected subgraph of $\overline{G}$ since $G\backslash V'$ is not connected. ■

Theorem 6: Let $G = (V, E)$ be a connected graph on $n \geq 4$ vertices, with vertex connectivity ν and radius r. Then $\nu \cdot r \leq 4\lfloor n/2 \rfloor - 4$ with equality if and only if G is K_5, or $G \cong K_n\backslash M$, where M is a perfect matching, if n is even; or G is a graph with $\delta = n-3$ and $\Delta \leq n-2$ such that $\overline{G}$ does not contain C_4.

Proof: We distinguish two cases.

Case 1: G is a graph with $r = 1$. In this case, from $\nu \leq n-1$ it follows that $\nu \cdot r \leq n-1$. If n is even, then $4\lfloor n/2 \rfloor - 4 = 2n-(4 > n-1)$ for $n \geq 4$, and the inequality holds. If n odd, then $4\lfloor n/2 \rfloor = 2n-6 \geq n-1$ for $n \geq 5$ with equality if and only if $n = 5$ and $G \cong K_5$.

Case 2: G is a graph with $r \geq 2$. Let v_0 be a vertex with minimum eccentricity and let v_r be a vertex of G such that $ecc(v_0) = d(v_0, v_r) = r$. Let $V_i = \{u \in V, d(u, v_0) = i\}$ for $i = 0, 1, \ldots, r$. Obviously, $V_i \cap V_j = \varnothing$ for all $i \neq j$. Also, V_i is a vertex-cut set for all $1 \leq i \leq r-1$, so $|V_i| \geq \nu$ for all $1 \leq i \leq r-1$. Furthermore, $|V_0| = 1$ and $|V_r| \geq 1$. Therefore, we have

$$n = \sum_{i=0}^{r} |V_i| \geq 2 + (r-1) \cdot \nu.$$

Thus,

$$\textbf{(2)} \qquad r \cdot \nu \leq n + \nu - 2.$$

If n is even, since $r \geq 2$ then $\nu \leq n-2$ and $r \cdot \nu \leq 2n-4 = 4\lfloor n/2 \rfloor - 4$, which is the desired bound. Equality holds if and only if $\nu = n-2$ and $r = 2$. Therefore, $\delta = \Delta = n-2$ and the corresponding graph is exactly $G\backslash M$, where M is a perfect matching.

If n is odd, from $r \geq 2$ it follows that each vertex is not connected to at least one vertex. Since n is odd there must be at least one vertex, say v_1, that is not connected to at least two vertices, say v_2 and v_3. Note that $V\backslash\{v_1, v_2, v_3\}$ is a vertex cut set, so that $\nu \leq n-3$. Using **(2)**, we have that $r \cdot \nu \leq 2n-5$ with possible equality if $\nu = n-3$; in which case ν is even (since n is odd) and then $r \cdot \nu$ is also even, whereas $2n-5$ is odd. Hence, $r \cdot \nu \leq 2n-6 = 4\lfloor n/2 \rfloor - 4$, which is the desired bound. Equality holds if and only if $\nu = n-3$ and $r = 2$, which implies that $n-3 \leq \delta \leq \Delta \leq n-2$. Note that the case $\delta = \Delta = n-2$ cannot hold since n and $n-2$ are odd. Thus, $\delta = n-3$ and $\Delta \leq n-2$. Therefore, characterization of the extremal graphs, in this case, follows from Lemma 5. ■

Now we consider the upper bound on $a \cdot r$.

Theorem 7: Let G be a connected graph on $n \geq 4$, $G \not\cong K_5$, with algebraic connectivity a and radius r. Then $a \cdot r \leq 4\lfloor n/2 \rfloor - 4$. That the bound is best possible is shown by K_4 and $K_n\backslash E$, where E is a perfect matching if n is even, and E is a set of connected components in $\overline{G}$ on at least two and at most three vertices each.

Proof: If $G \cong K_n$, then $a \cdot r = n \leq 4\lfloor n/2 \rfloor - 4$ for $n \geq 4$ with equality if and only if $n = 4$.

If $G \not\cong K_n$, according to **[11]**, $a \leq \delta$ and the inequality follows from Theorem 6.

To characterize the extremal graphs, let $G \not\cong K_4$ be a connected graph that satisfies the equality. Then, the bound in Theorem 6 is also attained. Thus, G is as described in Theorem 6 and it remains to prove that, indeed, for such a graph, $a = \nu$. If n is even, the result follows from Lemma 2. If n is odd and the connected components of $\overline{G}$ contain at least two and at most three vertices each, then G contains a subgraph that is a complete multipartite graph in which an independent set contains at least two and at most three vertices. Thus, using Lemma 2, $a = n - 3$, and the equality holds. Now, if $\overline{G}$ contains a component on at least four vertices, take four vertices, v_1, v_2, v_3, and v_4, from this component such that the induced subgraph H of $\overline{G}$ on $W = \{v_1, v_2, v_3, v_4\}$ is connected. Let G_1 be the induced subgraph of G on W. Note that G_1 is connected if and only if $G_1 \cong H \cong P_4$ (where P_4 is the path graph on four vertices); thus $a(G_1) < 1$. Consider the graph G' obtained from the complete graph on $V \backslash W$ and G_1 by inserting all possible edges between $V \backslash W$ and W. Because algebraic connectivity is non decreasing with respect to edge addition, then $a(G) \le a(G')$. According to **[11]**, $a(G') \le \min\{a(K_{n-4}) + |W|, a(G_1) + |V \backslash W|\}$. Therefore, $a(G) \le a(G') \le a(G_1) + n - 4 < n - 3$. ■

The following conjecture, concerning the upper bound on $\kappa \cdot r$, remains open.

Conjecture 8: Let $G = (V, E)$ be a connected graph on $n \ge 4$, with edge connectivity κ and radius r. Then $\kappa \cdot r \le 4\lfloor n/2 \rfloor - 4$, with equality if and only if G is K_5, or $G \cong K_n \backslash M$, where M is a perfect matching, if n is even, or G is a graph with $\delta = n - 3$ and $\Delta \le n - 2$ such that $\overline{G}$ does not contain C_4. □

4. Connectivity and Average Eccentricity

In this section, we prove the upper bounds on $\nu \cdot ecc$ and $a \cdot ecc$, and construct an infinite family of counter-examples for the conjectured upper bound on $\kappa \cdot ecc$.

Theorem 9: Let G be a connected graph on $n \ge 4$ vertices with vertex connectivity ν and average eccentricity ecc. Then

$$\nu \cdot ecc \le \begin{cases} 2n - 5 + 2/n & \text{if } n \text{ is odd} \\ 2n - 4 & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if $G \cong K_n \backslash M$, where M is a matching on $\lfloor n/2 \rfloor$ edges.

Proof: If n is even the bound follows from Theorem 1 and the fact that $ecc \le D$. Equality holds if and only if $\nu = n - 2$ and $ecc = 2$, in which case the corresponding graph is $K_n \backslash M$ where M is a perfect matching.

If n is odd and $\nu = 1$, then $G \cong K_n$ and $\nu \cdot ecc = n - 1 < 2n - 5 + 2/n$ for all $n \ge 5$.

If n is odd and $\nu \le n - 3$, then as seen in the proof of Theorem 1, $\nu \cdot D \le n - 2 + \nu = n - 5 < 2n - 5 + 2/n$. Thus $\nu \cdot ecc < 2n - 5 + 2/n$.

If n is odd and $\nu = n - 2$, then $\delta = n - 2$. Let t be the number of vertices of degree δ. We have

$$ecc = \frac{2t + n - t}{n} = \frac{n + t}{n}.$$

Thus

$$\nu \cdot ecc = (n - 2) \cdot \frac{n + t}{n} = n - 2 + \frac{n - 2}{n} t.$$

The last bound is maximum if and only if $t = n - 1$ and corresponds to $2n - 5 + 2/n$. It is easy to see that the corresponding extremal graph is $K_n \backslash M$ where M is a matching on $\lfloor n/2 \rfloor$ edges. ■

Theorem 10: Let G be a connected graph on $n \ge 3$ vertices, $G \ncong K_3$, with algebraic connectivity a and average eccentricity ecc. Then

$$a \cdot ecc \le \begin{cases} 2n - 5 + 2/n & \text{if } n \text{ is odd} \\ 2n - 4 & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if G is K_4 or $G \cong K_n \backslash M$, where M is a matching on $\lfloor n/2 \rfloor$ edges.

Proof: If $G \cong K_n$, then $a \cdot ecc = n$. If n is even, $a \cdot ecc = n \le 2n - 4$ with equality if and only if $n = 4$, which corresponds to $G \cong K_n$. If n is odd, $a \cdot ecc = n < 2n - 5 + 2/n$ for all $n \ge 5$.

Now, suppose that $G \ncong K_n$. In this case the inequality follows from Theorem 9 and the fact that $a \le \nu$ **[11]**.

To characterize the extremal graphs, it is easy to see that equality, in the present case, holds if and only if equality holds in Theorem 9 and $a = \nu$. Thus we must have $a = \nu = \delta = n-2$ with $ecc = 2$ if n is even, in which case the extremal graph corresponds to the complement of a perfect matching, or with $ecc = 2 - 1/n$, in which case the extremal graph corresponds to a complement of a matching on $(n-1)/2$ edges. ■

Conjecture 11: Let G be a connected graph on $n \geq 3$ vertices, $G \ncong K_3$, with edge connectivity κ and average eccentricity ecc. Then

$$\kappa \cdot ecc \leq \begin{cases} 2n-5+2/n & \text{if } n \text{ is odd} \\ 2n-4 & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if G is K_4 or $G \cong K_n \backslash M$, where M is a matching on $\lfloor n/2 \rfloor$ edges. □

This conjecture is not true. Consider the following sequential join

$$G_{k,x} \cong K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}} + K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}} + \ldots + K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}},$$

where there are $6x$ summands in the sum. It can easily be checked that $\kappa(G_{k,x}) = k$.
To disprove Conjecture 11, it is sufficient to prove that

$$\lim_{x \to \infty} \lim_{k \to \infty} \frac{ecc(G_{k,x}) \cdot k}{n(G_{k,x})} > 2.$$

Let l denote this double limit, then

$$l = \lim_{x,k \to \infty} \frac{k \frac{2(2\sqrt{k}3x + k(3x+1) + 2\sqrt{k}(3x+2) + 2\sqrt{k}(3x+3) + k(3((x+1)+1)) + \ldots + k(3(2x-1)+1) + 2\sqrt{k}(6x-1))}{2(2\sqrt{k}2x + xk)}}{2xk + 2 \cdot 2x \cdot 2\sqrt{k}}$$

$$= \lim_{x \to \infty} \frac{1}{2x} \cdot \frac{(3x+1) + (3(x+1)+1) + \ldots + (3(2x-1)+1)}{x}$$

$$= \lim_{x \to \infty} \frac{x(3x+1) + 3(1+2+\ldots+(x-1))}{2x^2} = \frac{3 + \frac{3}{2}}{2} = \frac{9}{4}.$$

Therefore Conjecture 11 is refuted.

5. Connectivity and Remoteness

In this section, we prove the upper bounds on $\nu \cdot \rho$ and $a \cdot \rho$. Then, we give an infinite family of counter-examples for the upper bound on $\kappa \cdot \rho$.

Theorem 12: Let G be a connected graph on $n \geq 2$ vertices with vertex connectivity ν and remoteness ρ. Then $\nu \cdot \rho \leq n-1$ with equality if and only if G is the complete graph K_n.

Proof: It is obvious that equality holds for K_n. Now suppose that $n \geq 3$ and $G \ncong K_n$, then $D \geq 2$. From the proof of Theorem 1, $D \leq 1 + (n-2)/\nu$. Also, it is easy to see that

$$\rho \leq \frac{\nu + (n-1-\nu) \cdot D}{n-1}.$$

Hence,

$$\nu \cdot \rho \leq \nu \cdot \frac{\nu + (n-1-\nu) \cdot D}{n-1} = \frac{\nu^2}{n-1} + \frac{\nu(n-\nu-1)}{n-1} \cdot D$$

$$\leq \frac{\nu^2}{n-1} + \frac{\nu}{n-1}(n-\nu-1) \cdot \left(\frac{n-2}{\nu} + 1\right)$$

$$= \frac{\nu^2}{n-1} + \frac{1}{n-1}(n-\nu-1) \cdot (n-2+\nu)$$

$$\begin{aligned}\nu \cdot \rho &= \frac{\nu^2}{n-1} + \frac{1}{n-1}(n^2 - 3n + 2 - \nu^2 + \nu) \\ &= \frac{1}{n-1}(n^2 - 3n + 2 + \nu) \le \frac{1}{n-1}(n^2 - 3n + 2 + n - 2) \\ &= \frac{1}{n-1}(n^2 - 2n) = n - 1 - \frac{1}{n-1} < n - 1.\end{aligned}$$

Thus the result follows. ■

Corollary 13: Let G be a connected graph on $n \ge 2$ vertices with algebraic connectivity a and remoteness ρ. If $G \cong K_n$, then $a \cdot \rho = n$. If $G \not\cong K_n$, then $a \cdot \rho \le n - 1 - 1/(n-1)$, with equality if and only if $G \cong K_n \backslash M$, where M is a matching.

Proof: It is obvious that if G is K_n, then $a \cdot \rho = n$.

Suppose that $G \not\cong K_n$. Then, as seen in the proof of Theorem 12, in this case we have

$$a \cdot \rho \le n - 1 - \frac{1}{n-1},$$

with equality if and only if $\nu = n - 2$ (and then, from Lemma 2, $a = n - 2$), that is, $G \cong K_n \backslash M$, where M is a matching. ■

Conjecture 14: Let G be a connected graph on $n \ge 2$ vertices with edge connectivity κ and remoteness ρ. Then $\kappa \cdot \rho \le n - 1$ with equality if and only if G is the complete graph K_n. □

This conjecture is not true. To provide a counter-example we use the same notation as in disproving Conjecture 4. Consider the following graph.

$$G_{k,x} \cong K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}} + K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}} + \ldots + K_{2\sqrt{k}} + K_k + K_{2\sqrt{k}},$$

where k is a perfect square and there are $6x$ summands. Note that $\kappa(G_{k,x}) = k$. To disprove the conjecture, it suffices to prove that

$$\lim_{x \to \infty} \lim_{k \to \infty} \frac{\rho(G_{k,x}) \cdot k}{n(G_{k,x})} > 1.$$

Let t denote this double limit, then

$$\begin{aligned} l &= \lim_{x,k \to \infty} k \cdot \left(\frac{(2\sqrt{k} - 1) + k + 2\sqrt{k} \cdot 2 + 2\sqrt{k} \cdot 3 + k(1+3) + \ldots + k(3(2x-1)+1) + 2\sqrt{k}(6x-1)}{2xk + 2 \cdot 2x\sqrt{k}} \right) \\ &= \lim_{x \to \infty} \frac{1 + (1+3) + \ldots + (3(2x-1)+1)}{4x^2} \\ &= \lim_{x \to \infty} \frac{2x + 3(1 + 2 + \ldots + (2x-1))}{2x^2} = 3 \cdot \frac{4/2}{4} = \frac{3}{2}. \end{aligned}$$

Therefore Conjecture 14 is refuted.

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[GTN LV:2] A COMPUTERIZED SYSTEM FOR GRAPH THEORY, ILLUSTRATED BY PARTIAL PROOFS FOR GRAPH-COLORING PROBLEMS[1]

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Abstract

The software system KBGRAPH, which supports graph theoretical proofs and the analysis of graph classes, is presented by developing partial proofs for two graph coloring problems. It is shown that Reed's Conjecture, which concerns an upper bound on the chromatic number, holds for some special classes of graphs; future approaches are briefly outlined. Another strengthening of Brooks' well-known upper bound is sketched. Details about the internal derivation strategies of the program and tools offered to the users are presented, as far as needed for an understanding of the subsequent sketch of a problem solving process. This article is written for a two-fold readership: readers who want a quick overview of the knowledge based system will find this in sections 1 – 4; for readers interested in more details of the software system, additional hints on its implementation, technical data, and the availability of the program are compiled in the last section.

1. A First Look at Open Problems

The knowledge-based system *KBGRAPH* is destined to support graph theoretical proofs and the analysis of graph classes. Instead of a boring sequence, describing one function of the program after the other, we want to let the reader participate in a step-by-step search for subsequent improvements, aiming at a proof of *Reed's Conjecture*. To date only partial proofs have been found. Typically, in the course of such a stepwise search, new relations between graph invariants are discovered, that are valid for all graphs and hence of independent interest.

Reed's Conjecture is an extension of a well-known upper bound on the *chromatic number*, $\chi(G)$, **[1]**.

Conjecture (Reed): For any graph G,

(1) $$\chi(G) \le \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil,$$

where $\Delta(G)$ denotes the maximum degree and $\omega(G)$ is the clique number of G. □

The chromatic number χ is a graph invariant that is connected with a great variety of other invariants, such that sharper bounds for χ may lead to an improved knowledge about other variables.

The following sketch shows how structural knowledge on the one hand, and inequalities stored within the knowledge based system, on the other hand, can be used to proceed toward partial proofs. The system *KBGRAPH* will frequently be "in the background". In any case, the following text is written such that no prior knowledge about the *KBGRAPH* system (nor of other such systems) is needed. Therefore, only a necessary short overview is given in the beginning (Section 3). Additional details for interested readers are placed in a separate section (Section 5).

[1] This paper is a revised and extended version of an article that appeared in *MATCH Commun. Math. Comput. Chem.*, **58**, 445–480 (2007), with kind permission.

As a by-product, new inequalities, partially connected with Reed's Conjecture, are listed separately (Section 4.5); in particular, sufficient conditions for a strengthening of Brooks' inequality such that $\chi \le \Delta$ is replaced by $\chi \le \Delta - 1$.

2. Notation

In this paper, *graph coloring* refers to coloring the vertices of graphs. All graphs considered are simple (finite, undirected, with no loop or multiple edge). As usual, G is a graph with p vertices and q edges, and $\overline{G}$ is the complement of G. Special graph classes are the *complete graphs* K_m, the *cycle graphs* C_m, and the *claw* $K_{1,3}$. Frequently occurring variables are the *minimum degree* δ and the *maximum degree* Δ, the *clique number* ω (that is, the number of vertices of the greatest clique), the *independence number* β_0, and the *vertex-cover number* α_0, the *vertex connectivity* κ, and the *edge connectivity* κ_1. We also write χ instead of $\chi(G)$, and so forth if this does not result in ambiguity.

3. Overview of the System *KBGRAPH*

The knowledge-based system *KBGRAPH* has two objectives:

- to analyze given classes of graphs, and
- to support proofs of graph theoretical hypotheses.

At present, the knowledge base consists of about 1,700 entries. Each such entry has the form of a known property of a graph invariant (e.g., $\omega \ge 2$) or of a relation between graph invariants, which may be unconditional (e.g., $\chi \ge \omega$) or conditional (*if…, then…*). Integer, real, and Boolean variables are permitted, as well as logical connectives (*and*, *or*, *not*). Each entry (except for some trivial cases) is equipped with a reference.

About 50 graph invariants are implemented, including all of the invariants mentioned in this paper. Material on other variables has been accumulated in the paper form, but not yet entered into the knowledge base.

A problem description consists of a finite list of *user-defined restrictions*. These have the same form as the knowledge-base entries: conditional or unconditional equations or inequalities; in practice, unconditional statements are more frequent. Any property of the considered class of graphs that is already known can be entered here as a user-defined restriction (see the example in Section 4.2). In those cases where an attempt of a mathematical proof is made, a problem description lists known properties of a hypothetical counterexample.

At the onset of the evaluation process, just after reading the user defined restrictions, an internal duplicate of the knowledge base is generated, and this is confronted with the user defined restrictions. The main evaluation process works in the usual mathematical style (forward chaining): known numerical and Boolean values are inserted, and the formulæ are simplified. In this way, the temporary duplicate of the knowledge base is permanently updated. As soon as the *if* part of a conditional statement is found to be true, this *if* part is deleted and the *then* part remains as an unconditional statement. If a *then* part turns out to be false, the negation of the *if* part is retained as a true statement. Transitivity of equality and of inequality relations is taken into account (e.g., $\kappa \le \kappa_1$ and $\kappa_1 \le \delta$ implies $\kappa \le \delta$). A special table is set up and permanently updated; this stores the currently best known numerical values for the lower and upper bounds of the numerical variables.

Within the general framework of forward chaining, a selection of specific techniques applied within the inference process can be sketched here:

- rounding in the case of integer variables: For example, $\chi < 9/2$ is replaced by $\chi \le 4$;
- deletion of formulæ that are inferior to other entries in the set of transformed formulæ;
- conclusions derived from the monotonicity of arithmetic functions: If, for example, $y = f(x)$ is a monotonically increasing function for $a \le x \le b$ ($a < b$), then it can be derived that $f(a) < f(b)$. (Monotonicity of a function can be recognized for linear or quadratic expressions and for functions of the type $y = c\log(x) + d$.)

Additional special methods of evaluation are described in Section 5.

When an inference run has ended, then generally concrete values for some variables have been identified, and improved bounds to some numerical variables have been obtained. These values and bounds are displayed to the user. Another part of the intermediate results consists of those knowledge-base entries that were altered by the evaluation runs. These formulæ can optionally be displayed on a screen, either completely or in a selective manner by use of a retrieval function. After any evaluation run, the user may enter further knowledge—possibly triggered by studying the recent results—and restart the evaluation. This may be repeated recursively as long as an improvement is found by the system.

If, in the case of an attempted proof, a contradiction is found, this means that the underlying class of graphs—that is, the class of hypothetical counterexamples—is empty, or, equivalently, that the hypothesis has been proved. This is signalled to the user, together with data about the formulæ that led to that contradiction.

KBGRAPH is consequently organized for interactive working. After the end of an evaluation run the user has the chance to:

- enter additional knowledge in the same style as the initial user defined restrictions;
- edit a single formula (e.g., to simplify an arithmetic expression by hand);
- enter the knowledge that some *if* part is true or some *then* part is false;
- tentatively insert numerical values for a numerical variable (in the case of a contradiction it is possible to increase a lower or to reduce an upper bound).

For each result, a *derivation tree* can be displayed, which shows how the result has been derived. The formula numbers lead to references from which formulæ of the original knowledge base were taken.

Some advanced evaluation techniques are also implemented; for example, working with *case distinctions*. For Boolean variables, the two alternatives can analyzed separately (e.g., *regular/not regular*); in the case of a numerical variable, the domain is decomposed into partial intervals (see an example in Section 4.2). There may be an identical improvement for the various cases that were detected in quite distinct methods of derivation. (For additional advanced evaluation techniques see Sections 4.2 and 5.2.)

A characteristic phenomenon appearing in evaluation processes can be dubbed *knowledge propagation*: improved knowledge about one variable is likely to advance the knowledge about other variables. Inspection of the ways in which surprising results come up suggests the term *crossword-puzzle phenomenon*. When a crossword puzzle is solved, a single new finding can trigger a *chain reaction* of additional new findings, such that finally entries for distant places will be found. Hence, any increase in a lower or decrease in an upper bound can be considered a chance for more progress. Furthermore, conditional formulas are activated as soon as a bound in a condition is reached.

The system supplies improved knowledge about the class of graphs considered. In particular, exact values for some graph invariants, sharper bounds for most other variables, and restrictions in the form of equations or inequalities to be fulfilled by graph invariants. If no proof is derived (which is the regular case for long-standing graph theoretical conjectures), then the new knowledge about properties of a counterexample may simplify the remaining task.

4. Reed's Conjecture

4.1. Problem Statement

Brooks **[1]** proved that for all connected graphs the inequality

(2) $\chi \le \Delta + 1$

holds, with equality if and only if G is a complete graph or an odd cycle (here only the case of connected graphs with $\chi \le \Delta$ is of interest). This was strengthened by Reed **[2][3]**, whose conjecture is stated above as **(1)**. Neither a proof nor a counterexample are known. This conjecture is trivial for $\omega = \Delta$ and for $\omega = \Delta + 1$. Reed **[2]** gave a proof for graphs with maximum degree $\Delta = p - 1$. A proof for all *line graphs* was presented at a conference in Berlin (June 2005, **[4]**); this proof stood in the context of a harder claim on multiedge graphs. A quick proof for line graphs—restricted to simple graphs—is obtained below as a by-product (Section 4.2, step 2). Contributions in two recent papers **[5][6]** are compiled below (Section 4.3).

4.2. A First Attempt with Reed's Conjecture

Unless otherwise stated, G denotes a counterexample to Reed's Conjecture. It is our goal to find additional and sharper constraints that the class of counterexamples need to fulfil.

Step 1: We can restrict our study to *color critical graphs* with chromatic number χ (χ-critical graphs). Every graph with chromatic number χ contains a χ-critical subgraph with the same number of vertices. If such a graph obeys **(1)**, then in any other graph generated from it by inserting edges, Δ and ω remain constant or increase, such that **(1)** continues to remain valid. Thus, we can make use of the known properties of color critical graphs. (The property *color critical* cannot be found automatically, but it is implemented in the system as a Boolean variable and is used if stated by the user.)

Step 2: By inserting $\omega = \chi$ and $\omega = \chi - 1$ into **(1)**, it turns out that **(1)** is fulfilled for these values. Hence, G must satisfy the constraint

(3) $\chi \geq \omega + 2$.

Some consequences of this property are stored in the system. Furthermore, it is known **[7]** that graphs obeying **(3)** must contain $K_{1,3}$ and/or K_5–e as an induced subgraph. These graphs are forbidden induced subgraphs for line graphs, and so it is quickly proved that **(1)** holds for line graphs.

Here, the restriction **(3)** was found by the user. In principle, it would be possible to start a program run without entering **(3)** as a user defined restriction—the program would be able to exclude $\omega = \chi$ and $\omega = \chi - 1$ in later phases. However, by doing so, the program run would be longer, and the intended demonstration would be rather clumsy. Furthermore, it should also be shown that the user's additional prior knowledge can be entered here.

Step 3: Next we check if $\chi = \omega + 2$ or $\chi = \omega - 2$ is possible. Since counterexamples are studied we have

$$\chi > \lceil (\Delta + \omega + 1)/2 \rceil$$

or equivalently

$$\chi > ((\Delta + \omega + 1 + \varepsilon)/2), \text{ and } \chi > \Delta - 1 + \varepsilon ,$$

with $\varepsilon = 1$ if $\Delta + \omega$ is even and $\varepsilon = 0$ if $\Delta + \omega$ is odd. The case $\varepsilon = 1$ must be excluded since here $\chi \leq \Delta$. Only the case $\varepsilon = 0$ remains. Then $\chi = \Delta$, and $\chi = \omega + 2$ implies $\chi \equiv \omega \pmod 2$, $\Delta \equiv \omega \pmod 2$, $\Delta + \omega \equiv 0 \pmod 2$, $\varepsilon = 1$, contrary to $\varepsilon = 0$. Hence, $\chi = \omega + 2$ is excluded, and with **(3)** we obtain

(4) $\chi \geq \omega + 3$.

This derivation cannot be accomplished by the system.

Step 4: As a next step we can compile the user defined restrictions:

R1: $\chi > \lceil (\Delta + \omega + 1)/2 \rceil$
R2: $\Delta \leq p - 2$
R3: color critical
R4: $\chi \geq \omega + 3$.

Here R1 is the negation of **(1)** since we are looking for a counterexample. R2 is a consequence of Reed's additional restriction as cited above. R3 was explained previously (Step 1), and R4 goes back to Step 3.

Step 5: With these user defined restrictions, a first evaluation run is started. Among the results only two points are worth reporting: A counterexample G is not completely multipartite, and it has $p \geq 11$. The latter finding is mainly due to a theorem by Nenov **[8]**: here, $\chi \geq 5$, and for $\omega \leq 3$ and $p \leq 10$ it follows that $\chi \leq 4$; for $\omega \geq 4$ with $\chi \geq 7$ the derivation is different.

Step 6: After the end of the first standard evaluation run, the special evaluation technique *working with case distinctions* is activated. We consider the complete case distinction {ω = 2, 3, 4, ≥ 5}, which means that the program will consecutively (but independently) handle the four cases:

$$\omega = 2 , \omega = 3 , \omega = 4 , \text{ and } \omega \geq 5 .$$

A selection of the results obtained is provided in the following table:

ω	= 2	= 3	= 4	≥ 5
$p \geq$	22	12	13	15
$q \geq$	47	34	43	58
$\chi \geq$	5	6	7	8
$\gamma \geq$	2	1	2	2

where γ is the *orientable genus*. First, note a little improvement from $p \geq 11$ to $p \geq 12$. The lower bound $p \geq 22$ for $\omega = 2$ can be traced back to a result by Jensen and Royle **[9]**: a graph with $\omega = 2$ and $\chi = 5$ has $p \geq 22$ (this lower bound also holds for $\chi \geq 6$).For the lower bounds to q, there are quite a lot of inequalities in the knowledge base, some of which also use parameters like ω and Δ; one of the most efficient lower bounds for q in the case of color critical graphs was found by Kostochka *et al.* (**[10]**, see Section 4.5, nr. 2). The lower bounds for χ follow from **(4)**.

In three of the four cases we have $\gamma \geq 2$. Before we proceed to the case with $\gamma \geq 1$, we study the functioning of the program module *working with case distinctions*. The same result, $\gamma \geq 2$, was derived in three different ways for the three cases (at the same time, it was noticed here that after a program run the derivation of each result was displayed). In this particular case, we could identify the underlying knowledge-base entries:

> **Case 1:** $\omega = 2$
>
> If $\gamma \leq 1$ and $\omega = 2$, then $\chi \leq 4$. **[11]**
>
> **Case 3:** $\omega = 4$
>
> If $\chi \geq 7$ and $\omega \leq 6$, then $\gamma \geq 2$. **[12]**
>
> **Case 4:** $\omega \geq 5$
>
> If $\gamma \leq 1$, then $\chi \leq 7$. **[11]**

Step 7: In view of the preliminary lower bounds for γ, the user may decide to handle the troublemaker—Case 2, with $\gamma \geq 1$ —separately. To this purpose, a new program run was started for Case 2 with the *hypothesis* $\gamma = 1$ as a new user defined restriction. This program run used all known results for this case and all restrictions defined previously, in particular, $\omega = 3$ and $\chi \geq 6$. Mainly on the basis of a formula by Dirac **[13]**, the program supplies $\chi = \delta = \Delta = 6$, such that G would be regular. However, according to Gould (**[14]**, p. 247), a color critical graph with $\delta \geq 3$ and $\delta = \chi$ cannot be regular. This contradiction, which is displayed to the user, excludes $\gamma = 1$, and so $\gamma \geq 2$ has been proved for this case.

To illustrate the flexibility of the system, we show an alternative proof for $\gamma \geq 2$ in Case 2: A *semi-automatic*, computer-assisted proof, which starts from the known facts $\omega = 3$ and $\chi \geq 6$. The retrieval function is activated and as a response to the query "χ and γ", about 20 formulæ containing χ and γ are displayed on the screen. Stimulated by a formula due to Thomassen **[12]**, the user can look up the original printed version. According to this source, most of the graphs with $\gamma = 1$ have $\chi \leq 5$, and hence can be ignored here. Two exceptional graphs have $\omega \geq 4$, contrary to $\omega = 3$. For the third of Thomassen's exceptional cases, one can combine the fact that here $p \geq 12$ with findings by Albertson and Hutchinson **[15]** resulting in the consequence that this last exceptional graph can also be omitted. Thus, we derive that G has $\gamma \geq 2$, or, in other words, that **(1)** holds for planar graphs and for toroidal graphs.

4.3. Contributions from the Theory of Graph Associations

The following inequalities, valid for all graphs, are taken from two recent papers **[5][6]**. Using the concept of graph associations, a theorem is found that permits us to derive new bounds for χ by choosing special types of induced subgraphs.

> **Definition:** Given a graph G and non-adjacent vertices a and b, we write $G/[a, b]$ for the graph obtained from G by associating (i.e., identifying) a and b into a single vertex $[a, b]$ and discarding any multiple edges. ■
>
> **Theorem 1:** Let G be a graph. Then, for any induced subgraph H of G

(5) $$\chi(G) \leq \chi(H) + \frac{p(G) + \omega(G) - p(H) - 1}{2}.$$ ■

There are two immediate applications. If G is connected and if the subgraph H is identified with a longest induced path P_m of G ($m \geq 3$, such that the diameter $d(P_m) = d(G) = m - 1 \geq 2$, then **(5)** leads to

(6) $$\chi(G) \leq \frac{p(G) + \omega(G) - d(G) + 2}{2}.$$

Next, suppose that G has $g \geq 5$ (where the *girth* g is the length of a shortest cycle) and take for H a subgraph induced by a shortest cycle together with its neighborhood, then

(7) $$\chi(G) \le \frac{p(G) - g(G)(\delta(G) - 1) + 7}{2}.$$

Theorem 2: Let G be a graph. Then

(8) $$\chi(G) \le \frac{p(G) + \omega(G) - \beta_0(G) + 1}{2},$$

or equivalently (with Gallai's relation $\alpha_0 + \beta_0 = p$)

(9) $$\chi(G) \le \frac{\omega(G) + \alpha_0(G) + 1}{2}.$$ ■

For triangle-free graphs it follows that

(10) $$\chi(G) \le \frac{p(G) - \Delta(G) + 3}{2}.$$

The following three inequalities are related to Reed's Conjecture. It was found that **(1)** holds for *decomposable graphs*; that is, for graphs G with a disconnected complement $\overline{G}$, such that G can be written as a *direct sum* $G = A + B$ (where $A + B$ means that each vertex of A is adjacent to each vertex of B). If G is a counterexample to **(1)**, then $\overline{G}$ has a perfect matching if p is even. For odd p, $\overline{G}$ is nearly matching-covered; that is, $\overline{G} - v$ has a perfect matching for any vertex v (see also **[16]**). Furthermore, $\overline{G}$ is bridgeless; that is, $\kappa_1(\overline{G}) \ge 2$.

Any counterexample to **(1)** must satisfy the inequalities

(11) $$\chi(G) \le \left\lceil \frac{p(G)}{2} \right\rceil,$$

(12) $$\Delta(G) \le p(G) - \sqrt{p(G) + 2\beta_0(G) + 1},$$

(13) $$\beta_0(G) \ge 3.$$

4.4. Example of an Advanced Evaluation Technique

As remarked previously, a proof for Reed's Conjecture has not yet been found. By experimenting with the program (and material from literature) we suggest that there are two graph classes for which a solution may be relatively easy: *Triangle free graphs* and *claw free graphs*. The following account is to show—based on an example—one of the advanced features of the program that the user can apply following a usual inference run.

Triangle free graphs are characterized by $\omega = 2$. Following an ordinary program run (for counterexamples to Reed's Conjecture), then including the new constraint $\omega = 2$, a run with a case distinction (*cf.* Section 4.2, Step 6) was used. The case distinction $\{g = 4, g \ge 5\}$ led to the result that $\gamma \ge 2$ for $g = 4$ and $\gamma \ge 3$ for $g \ge 5$. This suggests a test of whether $\gamma \ge 3$ could be proved for $g = 4$ also. The output of a new program run with the constraints $g = 4$ and $\gamma = 2$ consists of fixed values for nine numerical variables (e.g., $\chi = \Delta = 5$), whereas for all other numerical variables both lower and upper bounds are supplied.

Consequently, this is an ideal candidate for a program function called *automatic insertion*: For each integer variable that is constrained from both sides, all admissible values are inserted into the formulæ of the knowledge base and a contradiction close to a bound leads to an increase of a lower bound or a decrease of an upper bound (an extension to intervals bounded at one side and to real variables can only be mentioned here, see also Section 5.2). In the concrete case, for example, the interval $22 \le p \le 86$ was replaced by $22 \le p \le 56$, and $47 \le q \le 176$ was converted into $47 \le q \le 116$. In a similar way, inclusions for most of the other integer variables were strengthened.

Other advanced techniques of evaluation were also applied to Reed's Conjecture, but in this special case they did not lead to significant progress. Therefore, these techniques are handled in a general form in Section 5.2; a summary of partial results follows in Section 4.6.

4.5. Miscellaneous Inequalities

Related inequalities, vastly scattered in literature, should be registered here (with a short derivation or reference). The following formulæ (in nr. 1–3) were found by the retrieval function of *KBGRAPH* with the query

"χ and Δ". Some of them have immediate consequences for Reed's Conjecture. (The references are supplied by the system; a selection and some editing for the sake of easy reading were required.)

1. Brooks' well-known result, that all connected graphs (except for complete graphs and odd cycles) satisfy $\chi \le \Delta$, suggests to ask for conditions under which the stronger property

(14) $\chi \le \Delta - 1$

will hold. Some of the sufficient conditions are:

(a) If $\omega = 2$, then $\chi \le 2(\Delta + 2)/3 + 1$. **[17]**
$\rightarrow \omega = 2$ and $\Delta \ge 8$ implies **(14)**.

(b) If $\omega \le 3$, then $\chi \le 3(\Delta + 2)/4$. **[18]**
$\rightarrow \omega \le 3$ and $\Delta \ge 7$ implies **(14)**.

(c) If $\Delta \ge 7$ and $\omega \le (\Delta - 1)/2$, then $\chi \le \Delta - 1$. **[18]**

(d) If $\chi \ge \omega + 1$ and $\Delta > (p + 1)/2$ then $\chi \le \Delta - 1$.
If $\chi \ge \omega + 1$ and $\Delta \ge 9$ and $\Delta > p/2$, then $\chi \le \Delta - 1$. **[19]**

(e) If *G* has no C_4 (induced subgraph or not), then $\chi \le 2\Delta/3 + 2$. **[20]**
$\rightarrow$ If *G* has no C_4, then $\Delta \ge 7$ implies **(14)**.

(f) If *G* is color critical and $\kappa = 2$ with a cutset $\{u, v\}$,
then *u* and *v* are non-adjacent, and $\Delta \ge (3\chi - 5)/2$. **[14]** (p. 227)
$\rightarrow$ Under these conditions $\Delta \ge 6$ implies **(14)**.

(g) The Borodin–Kostochka Conjecture **[18]** claims:
If $\Delta \ge 9$ and $\omega \le \Delta - 1$, then $\chi \le \Delta - 1$.
Partial proofs were compiled in **[3]**. Recently this conjecture was proved for graphs containing a doubly critical edge; that is, an edge whose removal decreases χ by 2 **[21]**.

2. For color critical graphs, given χ and *p*, a good lower bound for *q* is required. At present, the best such bound is supplied by **[10]**, under the reservation that two classes of exceptional graphs are excluded. This restriction can be by expressed by the three *or*-connected properties:

If *G* is color critical and $4 \le \chi \le p - 2$ and ($2\chi \ne p + 1$ or $\beta_0 \ge 3$ or $\omega < (p - 1)/2$),
then $q \ge p(\chi - 1)/2 + \chi - 3$.

3. If *G* contains neither C_4 nor $2K_2$ as induced subgraphs, then $\chi(G) + \chi(\bar{G}) \ge p(G)$ and $\chi(G) \le \omega(G) + 1$ **[22]**. Thus, due to **(3)**, **(1)** also holds for this special class of graphs.

4. Every counterexample to **(1)** has $\Theta_0(G) \ge 5$, where $\Theta_0(G) = \chi(\bar{G})$ is the clique-to-vertex covering number. From **[23]** (Theorems 1 and 2) it follows that for all graphs

$\omega = 2$ implies $\Theta_0 \ge \chi$,

$\omega = 3$ implies $\Theta_0 \ge \chi - 1$.

From **(3)** we have $\chi \ge \omega + 3$, and in both cases $\Theta_0 \ge 5$ follows. For the case $\omega \ge 4$ and $\chi \ge 7$ a structure theorem by Dirac **[24]** can be used: Such a graph contains two vertices with $\chi - 1$ paths between them, where no two of these paths have an edge in common. Hence, a χ-critical graph ($\chi \ge 7$) cannot be covered by less than five cliques, with the order of these cliques bounded by $\chi - 3$. Here, at least five cliques are needed for a covering and the claim is proved.

5. As is shown above, Reed's Conjecture holds for all *line graphs*. This can be extended to a broader class of graphs. Line graphs have $\lambda_p \ge -2$ (where λ_p is the smallest adjacency eigenvalue) and there are exactly two classes of connected graphs sharing this property:

- generalized line graphs
- exceptional graphs.

For a definition of a *generalized line graph*, see for example **[25][26]**. Their relevant properties can best be found through a structural characterization given by Cvetkovic **[25]** (Theorem 2.2). A *generalized cocktail party graph* is a graph isomorphic with a clique with independent edges removed. The complement of a generalized cocktail party graph consists of isolated edges and vertices and so a generalized cocktail party graph

H has $\chi(H) = \omega(H)$. If G is a generalized line graph then its edges can be partitioned into generalized cocktail party graphs such that

Each vertex is in at most two generalized cocktail party graphs, and
two generalized cocktail party graphs have at most one common vertex.

Therefore, generalized line graphs do not obey **(4)**, and can be dropped here. Exceptional graphs are defined as connected graphs with $\lambda_p \geq -2$ that are neither line graphs nor generalized line graphs. With the explicit restriction to $\lambda_p > -2$ we find that there are 573 such graphs; they have at most 8 vertices **[26]** and, hence, can be ignored here. A counterexample to Reed's Conjecture has $\lambda_p \leq -2$. (The case of equality cannot be attacked with the present tools.)

4.6. Summary of Partial Results

Reed's Conjecture was proved for the following classes of graphs:

- Line graphs, generalized line graphs, and those exceptional graphs with $\lambda_p > -2$.
- Graphs with $\chi \leq \omega + 2$.
- Planar and toroidal graphs.
- Decomposable graphs.
- $\{C_4, 2K_2\}$-free graphs.

In any counterexample, variables have to satisfy the following lower bounds (a small selection): $p \geq 12$, $q \geq 34$, $\chi \geq 5$, $\delta \geq 4$, $\Delta \geq 5$, $\beta_0 \geq 3$, $\alpha_0 \geq 8$, $\gamma \geq 2$, $\Theta_0 \geq 5$, and $\lambda_p \leq -2$. Necessary structural properties of the complement $\overline{G}$ are compiled in Section 4.3; bounds on its invariants include: $\omega(\overline{G}) \geq 3$, $\chi(\overline{G}) \geq 5$, and $\Theta_0(\overline{G}) \geq 5$.

5. Additional Details about the System *KBGRAPH*

5.1. Starting Point and General Properties

The project *KBGRAPH* was started in 1985, stimulated by the appearance of a series of papers by Brigham and Dutton **[27]**–**[30]**. Their two compilations of relations between graph invariants **[29][30]** with overall 458 entries remain unparalleled; they formed the core of the knowledge base in the first version of *KBGRAPH*. The system has been independently developed further. Whereas forward chaining (see Section 3) has been maintained as the central evaluation strategy, *KBGRAPH* is now characterized by a quantitative increase (number of graph invariants and size of the knowledge bases) and by a series of novel features, mainly related to

- the user interface and the options for flexible post-processing,
- the advanced evaluation techniques (Section 5.2),
- the options for an external control of the inference process (Section 5.3).

At present, 51 graph invariants are implemented. The three knowledge bases include about 2,100 entries: About 1,700 in the *main knowledge base* and the rest in two *auxiliary knowledge bases* required for one of the special evaluation techniques (Section 5.2). According to individual requirements, graph invariants can be newly defined, cancelled, or renamed. The knowledge-base is permanently updated: Adding, deleting, or altering of entries is possible. From time to time a single entry is replaced by a stronger version.

Based upon forward chaining as the central inference method, the inference mechanism was programmed *ad hoc*, to adapt to the specific requirements of working with formulæ (no foreign software was used). Options for an external control of the inference are outlined in Section 5.3.

After the end of an inference run, the user can enter additional knowledge and start the inference process again, or apply one or the other of the *advanced evaluation techniques* (Section 5.2). This can be done repeatedly for as long as some progress is expected. After the end of each inference run, it is possible to display a derivation tree for each single result, and, in the case where more than one derivation method led to the same result, this fact is also disclosed to the user. Thus all findings can be checked and rewritten in the usual mathematical style.

5.2. Advanced Evaluation Techniques

Following an ordinary program run, the user may decide to use one of the following special techniques, all of which are optional:

- Working with case distinctions.
- Automatic insertion of values.
- Editing of a formula.
- Transition to a *related graph*.

Working With Case Distinctions: was already explained and illustrated in Section 4.2 (see Step 6). The user is free to define a decomposition of the domain of a variable (up to nine segments). No closed interval is required—a decomposition can have forms such as $\omega = \{2, 3, \geq 4\}$ or $\{ 3 \leq \Delta \leq 6 , \Delta \geq 7 \}$. The decomposition into sub-classes, sub-sub-classes, …, is supported by the system up to four hierarchy levels; within the same hierarchy level up to nine descendants of the same direct ancestor are permitted. If the same improvement is achieved for all subcases of the same case, then this new knowledge can be *reached upward* to the next common ancestor. The idea behind this—supported by experience—is the chance that the same improvement can be derived in different ways within the different subcases. Optionally, the system can make proposals for plausible case distinctions.

Automatic Insertion of Values: was exemplified in Section 4.4. It should be supplemented here that no closed interval is required. If an integer variable is bounded only from one side, then the tentative insertion of numerical values starts at that bound, and continues for as long as the formula just considered leads to a contradiction and thus makes it possible to narrow that bound. For the case of very large intervals and/or real variables, heuristic procedures exist that supply preliminary data to the user, who has to decide whether a proposed problem reduction seems plausible.

Editing: is possible for each of the formulæ that were transformed by an inference run. The user can

- simplify an arithmetic expression by hand,
- insert numerical or Boolean values for a variable,
- delete an *if* part if it is considered true,
- replace a *then* part by *false*,
- delete a formula (e.g., if it is recognized that an *if* part cannot be satisfied, or that an inequality is inferior to another one—the latter point is supported by the system).

Transition to a Related Graph: Some successful proofs in graph theory show that a transition from the given class of graphs to another class—called *related graphs* in short—may be advantageous. Such a transition can be defined by any unique unary graph transformation. There are formulæ that connect variables of the original graphs with variables of the related graphs—an example is provided by the transition to a complementary graph using theorems of the Nordhaus–Gaddum type or formulæ like $\omega(G) = \beta_0(\overline{G})$. After an ordinary inference run that yields new information on the original class of graphs, the user may switch over to a class of related graphs in order to start an inference process with respect to that second class. Then the new knowledge about the second class of graphs can be automatically transferred back to the original class of graph. Transitions to complementary graphs and to line graphs are implemented in the system. The required *interconnection knowledge bases* exist; these are the two *auxiliary knowledge bases* mentioned previously. The user is free to define additional types of derived graphs; in this case, of course, a corresponding interconnection knowledge base must be set up.

5.3. Options for External Control of the Inference Process

The essential options for an external influence on the inference process are:

- Masking.
- Ranking the variables.
- Working with or without a derivation tree.
- Partitioning the knowledge base.

Masking: Each graph invariant can be *masked*; that is, it will be treated as inexistent during the same session. This tool is mainly used if the user is sure that a certain variable will not contribute to the solution. Also, each single statement can be masked; thus, for example, it is possible to make an inference run with or without use of the four-color theorem.

Ranking of the Variables: A ranking, that is, a linear order, of all graph invariants is defined. In the case where an equality between two numerical variables is derived in the inference process, this ranking determines whether x will be substituted for y or *vice versa*. The ranking also has an influence on the order within output lists. The user can alter the ranking individually and store the new ranking for future use.

Working With or Without a Derivation Tree: The user can decide whether or not a derivation tree is to be built up during an inference run. The derivation tree is required if the user later wants to obtain information about the way a certain result has been derived. Working without the derivation tree will reduce the program runtime.

Partitioning the Knowledge Base: In view of the extended knowledge bases, strategies can be recommended to speed up the inference process. The main knowledge base is partitioned in the following way. Each of its entries is assigned to one of three subsets whose members may be named *very important*, *important*, or *less important*. At the outset, only the very important class statements are used; in later inference runs those in the important class are included, until finally all entries in the knowledge base are active. In this way, some useful intermediate results can be achieved in earlier inference runs, with the consequence that many expressions can be simplified rather soon. Four different strategies for a partitioning of the knowledge base were empirically tested. It was found that in most cases this technique leads to a considerable reduction in computing time. The user may choose among these four strategies—in any case, one of them is predefined as a standard (according to the empirical results). For details consult the paper **[31]**.

5.4. Technical Details, References, Availability

Implementation of the system started in 1985 and continued until 2000. Since 2000, no further revision of the program was possible, but the knowledge bases are permanently updated. Due to side conditions about the year 1985 (students' knowledge and equipment, also administrative rules), the system was programmed in PASCAL (in the final stage about 30,000 lines of code) and on the basis of MS-DOS; the menus are in German. A transfer to modern computers is possible, and has already been successfully performed.

Additional information, for example, about examples of application, criteria for the selection of graph invariants and formulæ, parameter dependence of required computer runtime, and strategies to speed up computer runs with large knowledge bases, can be found in two papers **[32][33]** that contain many references. The executable program is freely available (email: **<t4141ax@mail.lrz-muenchen.de>**), as well as the published and unpublished expertise acquired in many years of practical work with the system.

6. Recent Little Steps and a Short Outlook

Small improvements that were found after finishing the main part of this paper are reported here without proof. Every counterexample to Reed's Conjecture must fulfil: $p \geq 14$, $q \geq 34$, $\alpha_0 \geq 10$, and $\Delta \leq p - 7$.

Future research on Reed's Conjecture may start from the abundant literature on color critical graphs, of which only a small proportion has been used to date. Another promising approach could be based on the complements of possible counterexamples—some structural properties of these complementary graphs are compiled here.

Practical use of the system *KBGRAPH* continues. The knowledge bases are permanently updated; but nevertheless the system should be reprogrammed totally from the beginning, free from restrictions imposed by earlier hardware, based upon a modern programming language and operating system, and exploiting the expertise accumulated over the years with respect to design, updating, and practical work.

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[GTN LV:3] A NOTE ON EDGE DEGREE WEIGHTED SUMS OF A GRAPH

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Abstract

Let $G = (V, E)$ be a simple graph with p vertices and q edges. Let each edge $e = uv$ of G be weighted $w(e) = \deg(u) + \deg(v)$. Define the edge degree weighted sum of G to be the sum of the edge weights of G, $w(G) = \sum_{e \in E(G)} w(e)$. In this paper we investigate properties of $w(G)$ and determine its value for various graphs.

Let $G = (V, E)$ be a simple graph with p vertices and q edges.

Definition 1: Let each edge $e = uv$ in E be weighted with the sum of the degrees at each endvertex so that $w(e) = \deg(u) + \deg(v)$ is its *weight*. The *edge weighted degree sum* of G is defined to be the sum of the edge weights of G and is denoted by $w(G) = \sum_{e \in E} w(e)$. (Note: if G is an empty graph, we define $w(G) = 0$.) ■

We use the notation P_p, W_p, K_p, and C_p to denote, respectively, the *path graph*, *wheel*, *complete graph*, and *cycle graph* with p vertices; $K_{1,p}$ the *star graph* with $p + 1$ vertices, and $K_{m,n}$ the *complete bipartite graph* with $m + n$ vertices (see **[1]**).

Theorem 1:

(1) If $G \cong P_p$ ($p > 1$), then $w(G) = 4p - 6$; $w(e) = 4$ for each non-pendant edge e in E, and $w(e) = 3$ for each of the two pendant edges.

(2) If $G \cong W_p$ ($p > 3$), then $w(G) = p(p + 7) - 8$; $w(e) = p + 2$ for each inner edge e in E, and $w(e) = 6$ for each outer edge e in E.

(3) If $G \cong K_{1,p}$ ($p > 0$), then $w(G) = p(p + 1)$; $w(e) = p + 1$ for each edge e in E.

(4) If $G \cong K_{m,n}$, then $w(G) = mn(m + n)$; $w(e) = m + n$ for each edge e in E.

Proof: Follows directly from the definition. ■

Theorem 2: If $G = (V, E)$, then $w(G) = \sum_{v \in V} (\deg(v))^2$.

Proof: Each vertex v in V contributes $\deg(v)$ to the weights of $\deg(v)$ edges and thus contributes $(\deg(v))^2$ to $w(G)$. ■

Corollary 2.1: For any graph $G = (V, E)$, then $w(G)$ is even.

Proof: Since for any graph the number of vertices of odd degree is even, then $w(G) = \sum_{v \in V} (\deg(v))^2$ is also even. ■

Corollary 2.2: The number of edges with odd weight is even. ■

Corollary 2.3: If G is regular of degree r, then $w(G) = pr^2$ and $w(e) = 2r$ for each edge e in E.

Proof: Each edge in E is assigned the same value $w(e) = 2r$ since each incident vertex has the same degree. Since each vertex has degree r, it follows from Theorem 2 that $w(G) = pr^2$. ■

Corollary 2.4:

(1) If $G \cong K_p$ $(p > 0)$, then $w(G) = p(p-1)^2$ and $w(e) = 2(p-1)$ for each edge *e*.

(2) If $G \cong C_p$ $(p > 2)$, then $w(G) = 4p$ and $w(e) = 4$ for each edge *e*.

(3) If $G \cong \overline{K_p}$ $(p > 0)$ is the empty graph with *p* vertices, then $w(G) = 0$. ■

Corollary 2.5: For all graphs *G* with *p* vertices, $\max\{w(G)\} = p(p-1)^2$.
$w(G) = p(p-1)^2$ if and only if $G \cong K_p$. ■

Theorem 3:

(1) $w(G+e) = w(G) + 2(\deg(u) + \deg(v) + 1)$, when $e \notin E(G)$.

(2) $w(G-e) = w(G) - 2(\deg(u) + \deg(v) - 1)$, when $e \in E(G)$. ■

Corollary 3.1:

(1.1) $w(C_p + e) = w(C_p) + 2(2+2+1) = 4p + 10$.

(1.2) If *e* connects the pendant vertices of P_p, then
$w(P_p + e) = w(P_p) + 2(1+1+1) = 4p - 6 + 6 = w(C_p)$

(2.1) $w(C_p - e) = w(C_p) - 2(2+2-1) = 4p - 6 = w(P_p)$.

(2.2) $w(K_p - e) = w(K_p) - 2\{(p-1) + (p-1) - 1\}$
$= p(p-1)^2 - (4p-6) = p^3 - 2p^2 - 3p + 6.$

■

Theorem 4:

(1) Given an even number $x \geq 0$, there exists a graph *G* such that $w(G) = x$.

(2) Given an even number $x \geq 0$, there exists a connected graph *G* such that $w(G) = x$ if and only if *x* is neither 4 nor 8.

Proof:

(1) If $x = 0$, let $G \cong K_1$, else let *G* be a union of $x/2$ disjoint edges on $v > 0$ vertices.

(2) If $x = 0$, let $G \cong K_1$. If $x = 2$, then let $G \cong K_2$. If $v = 4$, then there is no connected graph since there must be at least two adjacent edges that would add at least 6 to $w(G)$. If $x = 6$, then let $G \cong P_3$. If $x = 8$, then there is no connected graph since there must be at least three edges that would form a path P_4 or a $K_{1,3}$ and this would add at least 10 to $w(G)$. If $x \geq 10$, then let *G* be an apposite path graph or cycle. ■

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[GTN LV:4]

ESTIMATING THE DISTANCE ENERGY OF GRAPHS

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Abstract

The distance energy E_D of a graph G is defined as the sum of the absolute values of the eigenvalues of the distance matrix of G. Recently, bounds on E_D for graphs of diameter 2 were determined. In this paper we obtain bounds on E_D that are valid for any connected graph, thus generalizing and extending previous results.

1. Introduction

In this paper we are concerned with simple graphs; that is, graphs with no loop, multiple or directed edge. Let G be such a graph, possessing n vertices and m edges. We say that G is an (n, m)-graph.

Let the graph G be connected and let its vertices be labelled $v_1, v_2, \ldots, v_n$. The *distance matrix* of G is defined to be the square matrix $\mathbf{D} = \mathbf{D}(G) = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G **[1]**. The *eigenvalues* of the distance matrix are denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and are said to be the D-eigenvalues of G. Since the distance matrix is symmetric, its eigenvalues are real and can be ordered: $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. Since all diagonal elements of $\mathbf{D}$ are equal to zero,

$$\text{(1)} \qquad \sum_{i=1}^{n} \mu_i = 0.$$

The characteristic polynomial and eigenvalues of the distance matrix were much studied in the past **[2]**–**[7]**. Recently the *distance energy* $E_D = E_D(G)$ of a graph G has been defined **[8]** as

(2) $$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i|.$$

The form of Equation **(2)** has (deliberately) been chosen so as to be fully analogous to the definition of graph energy **[9]**–**[11]**

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the ordinary graph eigenvalues **[12]**; that is, the eigenvalues of the adjacency matrix $\mathbf{A}(G)$. Recall that in the past few years, the graph energy $E(G)$ has been extensively studied in the mathematics **[13]**–**[19]** and mathematico-chemical literature **[20]**–**[26]**.

We approach the study of the distance energy by making the following observation. In the theory of ordinary graph energy **[10][11]** there are numerous results (lower and upper bounds, approximate formulas, and so forth) relating $E(G)$ with the parameters n and m. In most cases, the parameter m enters the theory via the well known relation **[12]**

(3) $$\sum_{i=1}^{n} (\lambda_i)^2 = 2m.$$

The distance-spectral analog of Equation **(3)** is

(4) $$\sum_{i=1}^{n} (\mu_i)^2 = 2M,$$

where

$$M = \sum_{i<j} (d_{ij})^2.$$

Equation **(4)** is easily deduced:

$$\sum_{i=1}^{n} (\mu_i)^2 = Tr\mathbf{D}^2 = \sum_{i=1}^{n} [\mathbf{D}^2]_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} (d_{ij})^2 = 2\sum_{i<j} (d_{ij})^2 .$$

In a recent paper **[8]** Indulal, Vijaykumar, and Gutman reported lower and upper bounds for the distance energy of graphs whose diameter (that is, maximal distance between vertices) does not exceed two. In this paper we obtain bounds for the distance energy of arbitrary connected (n, m)-graphs that generalize the results obtained in **[8]**.

In an (n, m)-graph of diameter less than or equal to two, there are m pairs of vertices at distance one, whereas the remaining $\binom{n}{2} - m$ pairs of vertices are at distance two. Therefore,

$$M = m \cdot 1^2 + \left[\binom{n}{2} - m\right] \cdot 2^2 = 2n^2 - 2n - 3m.$$

2. Bounds on the Distance Energy

Theorem 1: Let G be a connected (n, m)-graph and let Δ be the absolute value of the determinant of the distance matrix $\mathbf{D}(G)$. Then

(5) $$\sqrt{2M + n(n-1)\Delta^{2/n}} \le E_D(G) \le \sqrt{2Mn}.$$

Proof:

(1) Lower bound: By the definition of distance energy and Equation **(4)**,

(6)
$$E_D(G)^2 = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 = \sum_{i=1}^{n} (\mu_i)^2 + 2\sum_{i<j} |\mu_i||\mu_j|$$
$$= 2M + 2\sum_{i<j} |\mu_i||\mu_j| = 2M + \sum_{i \neq j} |\mu_i||\mu_j|.$$

Since for nonnegative numbers the arithmetic mean is not smaller than the geometric mean,

$$\frac{1}{n(n-1)}\sum_{i \neq j} |\mu_i||\mu_j| \geq \left(\prod_{i \neq j} |\mu_i||\mu_j|\right)^{\frac{1}{n(n-1)}} = \left(\prod_{i=1}^{n} |\mu_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$
$$= \prod_{i=1}^{n} |\mu_i|^{2/n} = \Delta^{2/n}.$$

Therefore,

(7)
$$\sum_{i \neq j} |\mu_i||\mu_j| \geq n(n-1)\Delta^{2/n}.$$

Combining **(6)** and **(7)** we obtain the lower bound.

(2) Upper bound: Consider the quantity X

$$X = \sum_{i=1}^{n}\sum_{j=1}^{n} (|\mu_i| - |\mu_j|)^2,$$

the value of which is evidently non-negative. By direct expansion,

$$X = n\sum_{i=1}^{2} (\mu_i)^2 + n\sum_{j=1}^{2} (\mu_j)^2 - 2\left(\sum_{i=1}^{n} |\mu_i|\right)\left(\sum_{j=1}^{n} |\mu_j|\right),$$

which, in view of Equations **(2)** and **(4)** yields

$$X = 2nM + 2nM - 2(E_D)^2$$

and the upper bound follows from $X \geq 0$. ■

By substituting into the estimates **(5)** $M = 2n^2 - 2n - 3m$ we obtain a result for graphs of diameter 2:

Corollary 1.1: If an n-vertex graph G has diameter 1; that is, if $G \cong K_n$, then $E_D(G) = E(G) = 2n - 2$. If G is an (n, m)-graph of diameter 2, then

$$\sqrt{4n^2 - 4n - 6m + n(n-1)\Delta^{2/n}} \leq E_D(G) \leq \sqrt{4n^3 - 4n^2 - 6mn}.$$ ■

The above inequalities were reported in **[8]**.

For any n-vertex tree T **[2]**,

$$\det\mathbf{D}(T) = (-1)^{n-1}(n-1)2^{n-2},$$

which leads to:

Corollary 1.2: If T is an n-vertex tree, then

$$E_D(G) \geq \sqrt{2M + 4n(n-1)\left(\frac{n-1}{4}\right)^{1/n}}.$$ ■

Theorem 2: Let G be a connected (n, m)-graph and let M be the quantity defined by Equation **(4)**. Then

$$2\sqrt{M} \le E_D(G) \le \sqrt{M(1+\sqrt{8M+1})}. \quad \textbf{(8)}$$

Proof:

(1) Lower bound: From **(1)** and **(4)** we obtain

$$\left(\sum_{i=1}^{n} \mu_i\right)^2 = 2M + 2\sum_{i<j} \mu_i\mu_j = 0,$$

that is,

$$\sum_{i<j} \mu_i\mu_j = -M.$$

Now,

$$(E_D)^2 = \sum_{i=1} (\mu_i)^2 + 2\sum_{i<j} |\mu_i||\mu_j|$$
$$\ge 2M + \left|\sum_{i<j} \mu_i\mu_j\right| = 2M + 2|-M| = 4M$$

and the lower bound follows.

(2) Upper bound: The minimal possible value for M of an (n, m)-graph is $\binom{n}{2}$, attained for the complete graph, in which any two vertices are at unit distance. From $\binom{n}{2} \le M$ it follows that

$$n \le \frac{1}{2}(1+\sqrt{8M+1}).$$

By combining this inequality with the upper bound in **(5)** we obtain the upper bound in **(8)**. ■

Theorem 3: Let G be a connected (n, m)-graph. Then

$$\sqrt{2n(n-1)} \le E_D(G) \le \sqrt{\frac{n^3(n^2-1)}{6}}. \quad \textbf{(9)}$$

Proof: As already mentioned, the minimal value of M is $n(n-1)/2$. When this is substituted into the lower bound in **(8)** we obtain the lower bound in **(9)**.

From results obtained elsewhere **[27][28]** we know that the maximal possible value of M for a connected (n, m)-graph is equal to $n^2(n^2-1)/12$, and is attained for the n-vertex path. The upper bound in **(9)** then follows from the upper bound in **(5)**. ■

3. Discussion

At this time it is difficult to see how good the estimates given in Theorems 1–3 are. What is certain is that these bounds are not best possible. It would be of some value to find out which connected n-vertex graphs have the smallest and greatest distance-energy. In this connection, it looks plausible to state the following:

Conjecture 1: The complete graph K_n is the connected n-vertex graph with smallest distance energy (equal to $2(n-1)$). □

We do not dare to state a conjecture about the graph with greatest distance energy, but the path graph P_n certainly deserves to be seriously considered as a candidate.

As a concluding remark we mention that the inequalities stated here as Theorems 1, 2, and 3, have analogies in the theory of ordinary graph energy. The "pair" of Theorem 1 is quite similar to **(8)**, viz.:

$$\sqrt{2m + n(n-1)|\det\mathbf{A}(G)|^{2/n}} \le E(G) \le \sqrt{2mn},$$

and was obtained long time ago **[29]**. The "pairs" of Theorems 2 and 3 read **[30][31]**:

$$2\sqrt{m} \leq E(G) \leq 2m$$

and

$$2\sqrt{n-1} \leq E(G) \leq \frac{n}{2}(\sqrt{n}+1)$$

and have forms that significantly differ from **(8)** and **(9)**.

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[GTN LV:5] VERTEX MEASURABLE GRAPHS

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Abstract

Let G be a graph and I be a σ-field of induced subgraphs of G. A non-negative extended real valued countably additive graph function μ_v on $I \neq \varnothing$, is called a vertex measure on I. The ordered triplet (G, I, μ_v) is said to be a vertex measure space.

1. Introduction

The authors introduced the concept of *edge measurability* of a graph and proved some results related to this concept elsewhere **[1][2]**. In this paper we study the vertex analog of the concept and develop results related to this concept.

For basic concepts and terminology not given here we follow **[3]** for graph theory and **[4]** for measure theory. First, recall some basic definitions in graph theory.

Definition 1.1: A graph G with p vertices and q edges is called a (p,q) graph, where p and q are, respectively, known as the *order* and the *size* of the graph G. ■

Definition 1.2: A (p,q) graph with $p \neq 0$ and $q = 0$ is called an *empty graph* and is denoted by φ. ■

Definition 1.3: A (p,q) graph with $p = q = 0$ is called a *null graph* and is denoted by ϕ. ■

Definition 1.4: Let G be a graph and H be a subgraph of G. The *edge complement* of H in G is the subgraph of G obtained by deleting all the edges of H from G and is denoted by H'. ■

To define vertex measurability, we first need the following:

Definition 1.5: Let $G = (V, E)$ be a graph and $H = (V^1, E^1)$ be a subgraph of G. The *vertex complement* of H in G is denoted by $H^c_{V_G}$ and is defined as the subgraph obtained from G by deleting all the vertices of H. That is, $H^c_{V_G} = G[V - V^1]$. Hereafter, we use H^c rather than $H^c_{V_G}$. ■

Example: For the graph G shown in Figure 1 and its subgraph H shown in Figure 2, the vertex complement is shown in Figure 3.

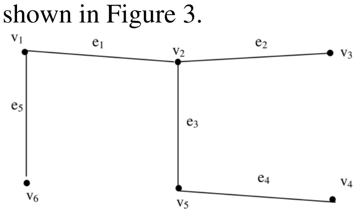

Figure 1: *G*.

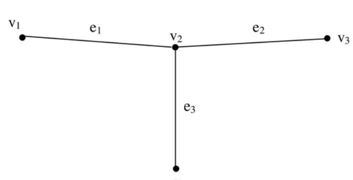

Figure 2: *H*.

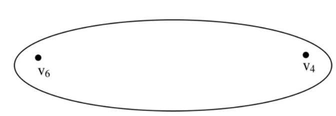

Figure 3: H^c.

We make the following observations about the vertex complement of a subgraph H of a graph G.

Observations:

(1) $V(H) \subseteq V(G)$, $V(H^c) \subseteq V(G)$, $V(H) \cup V(H^c) = V(G)$.

(2) H and H^c are vertex disjoint subgraphs of G.

(3) H and H^c are edge disjoint subgraphs of G.

(4) $(H^c)^c$ is not always equal to H.

(5) $H \cup H^c$ need not be equal to G. ■

Observation (4) tells that the structure property is not satisfied.

Example: Consider the graph G shown in Figure 4 and its subgraph H_1 shown in Figure 5, the vertex complement H_1^c of H_1 is shown in Figure 6.

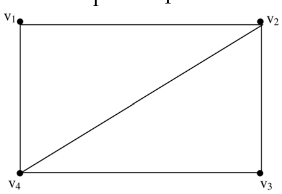

Figure 4

Figure 5

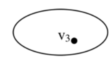

Figure 6

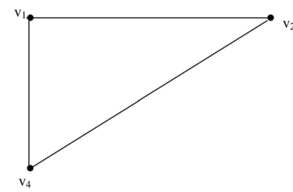

Figure 7

The graph $(H_1^c)^c$ is displayed in Figure 7. From Figures 5 and 7 it is clear that $(H_1^c)^c \neq H_1$. Hence, the structure property is not satisfied. Additionally, as we observed, $H \cup H^c$ need not be equal to G.

Example: Consider the subgraph H illustrated in Figure 2 and its vertex complement shown in Figure 3. Their union is shown in Figure 8. It is clear that $H \cup H^c \neq G$. There is a subgraph, displayed in Figure 9, that is not found in the union. Such a subgraph is called a *hidden subgraph* of G related to H and is denoted by H^h.

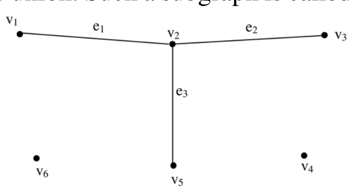

Figure 8

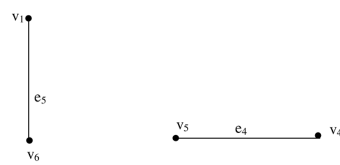

Figure 9

Remark: Let G be a graph and let H be a subgraph of G. The hidden subgraphs related to H and to H^c are the same.

2. Vertex Measurable Graphs

In this section we first make some observations about H^h.

Observations: Let G be a graph and H be a subgraph of G. Let H^c be the vertex complement of H in G and let H^h be the hidden subgraph related to H. Then

(1) $V(H) \cup V(H^c) \cup V(H^h) = V(H) \cup V(H^c) = V(G)$.

(2) $E(H) \cup E(H^c) \cup E(H^h) = E(G)$.

(3) H, H^h, and H^c are edge disjoint subgraphs of G. ■

This leads to the following definition.

Definition 2.1: Let $G = (V, E)$ be a graph and for $S_1 \subset V$ and $S_2 \subset V$, let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be two induced subgraphs of G. The *induced union* of $\langle S_1 \rangle$ and $\langle S_2 \rangle$ is the induced subgraph $\langle S_1 \cup S_2 \rangle$. ■

For convenience, let $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$. We denote $\langle S_1 \cup S_2 \rangle$ by $H_1 \uplus H_2$. In general,

$$\langle \bigcup_{i=1}^{n} S_i \rangle = \biguplus_{i=1}^{n} H_i \text{, where } S_i \subset V \text{ for } i = 1, 2, \ldots, n .$$

Example: Consider the graph G shown in Figure 10. Here $V = \{v_1, v_2, \ldots, v_8\}$. Let $S_1 = \{v_1, v_2, v_3, v_4\}$ and then $H_1 = \langle S_1 \rangle$ is as shown in Figure 11. Let $S_2 = \{v_5, v_6, v_7\}$ and $H_2 = \langle S_2 \rangle$ is as shown in Figure 12. Finally, $H_1 \mathbin{\overset{\times}{\cup}} H_2 = \langle S_1 \cup S_2 \rangle$ is as illustrated in Figure 13.

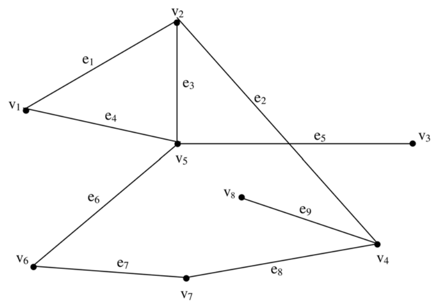

Figure 10

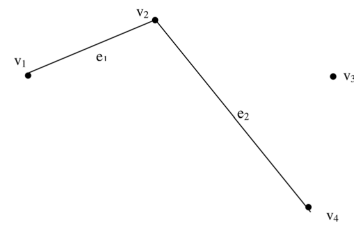

Figure 11

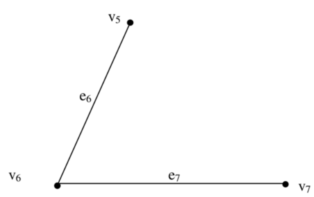

Figure 12

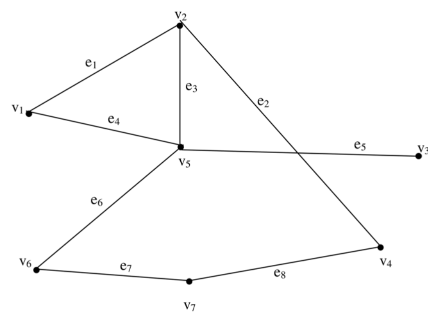

Figure 13

Observations: Let $G = (V, E)$ be a graph.

1. For $S_1, S_2 \subset V$, if $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$ are two vertex induced subgraphs of G, then $H_1 \cup H_2 \subseteq H_1 \mathbin{\overset{\times}{\cup}} H_2$.

2. $H_1 \cap H_2 = \langle S_1 \cap S_2 \rangle$; that is, $\langle S_1 \rangle \cap \langle S_2 \rangle = \langle S_1 \cap S_2 \rangle$.

3. If $H_1 = \langle S_1 \rangle$, where $S_1 \subset V$, then H_1^c, the vertex complement of H_1 in G, is the vertex induced subgraph $\langle V - S_1 \rangle$.

4. If $H_1 = \langle S_1 \rangle$, where $S_1 \subset V$, then $V(H_1 \mathbin{\overset{\times}{\cup}} H_1^c) = V(G)$ and $E(H_1 \mathbin{\overset{\times}{\cup}} H_1^c) = E(G)$. $H_1 \mathbin{\overset{\times}{\cup}} H_1^c = G$ and $H_1 \cap H_1^c = \varnothing$.

5. If $H_1 = \langle S_1 \rangle$, where $S_1 \subset V$, then $H_1^c = \langle V - S_1 \rangle$.
Therefore, $(H_1^c)^c = \langle V - (V - S_1) \rangle = \langle S_1 \rangle$. ■

Note: H_1 and H_1^c are not only edge disjoint but also vertex disjoint subgraphs of G.

By Properties 4 and 5 one can easily confirm preservation of the structure property.

Theorem 2.2: Let $G = (V, E)$ be a graph. For $S_1 \subset V$ and $S_2 \subset V$, let $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$ be two induced subgraphs of G. Then

(1) $(H_1 \mathbin{\overset{\times}{\cup}} H_2)^c = H_1^c \cap H_2^c$.

(2) $(H_1 \cap H_2)^c = H_1^c \mathbin{\overset{\times}{\cup}} H_2^c$.

Proof:

$$H_1 = \langle S_1 \rangle \Rightarrow H_1^c = \langle V - S_1 \rangle \text{ and } H_2 = \langle S_2 \rangle \Rightarrow H_2^c = \langle V - S_2 \rangle .$$

Therefore, $H_1 \mathbin{\overset{\times}{\cup}} H_2 = \langle S_1 \cup S_2 \rangle$. This implies:

(1) $(H_1 \mathbin{\overset{\times}{\cup}} H_2)^c = \langle V - (S_1 \cup S_2) \rangle$.

Now,

(2) $H_1^c \cap H_2^c = \langle V - S_1 \rangle \cap \langle V - S_2 \rangle = \langle V - (S_1 \cup S_2) \rangle$.

From **(1)** and **(2)** it follows that

$$(H_1 \uplus H_2)^c = H_1^c \cap H_2^c .$$

Since

$$H_1 \cap H_2 = \langle S_1 \rangle \cap \langle S_2 \rangle = \langle S_1 \cap S_2 \rangle ,$$

then

(3) $(H_1 \cap H_2)^c = \langle V - (S_1 \cap S_2) \rangle$.

Furthermore,

$$H_1^c \uplus H_2^c = \langle V - S_1 \rangle \cup \langle V - S_2 \rangle = \langle V - (S_1 \cap S_2) \rangle .$$

From **(3)** and **(4)** it follows that

$$(H_1 \cap H_2)^c = H_1^c \uplus H_2^c . \quad \blacksquare$$

Theorem 2.2 can be generalized as follows.

Corollary 2.3: If $H_1 = \langle S_1 \rangle$, $H_2 = \langle S_2 \rangle$, …, $H_n = \langle S_n \rangle$ are induced subgraphs of a graph $G = (V, E)$, then

(1) $\left(\biguplus_{i=1}^{n} H_i \right)^c = \bigcap_{i=1}^{n} H_i^c$ and (2) $\left(\bigcap_{i=1}^{n} H_i \right)^c = \biguplus_{i=1}^{n} H_i^c$. ■

Lemma 2.4: Let G be a graph and let I be a collection of induced subgraphs of G, together with the empty graph ϕ. I is a field if and only if

(1) $G \in I$,

(2) $H \in I \Rightarrow H^c \in I$ for each H, and

(3) $H_1, H_2, \ldots, H_n \in I \Rightarrow \biguplus_{i=1}^{n} H_i \in I$ and $\bigcap_{i=1}^{n} H_i \in I$. ■

Lemma 2.5: If condition (3) in Lemma 2.4 is replaced by a countable induced union and a countable intersection, then I is a σ-field of G. ■

We note that G may be infinite, and then if $H_1, H_2, \ldots \in I$, then $\biguplus_{i=1}^{\infty} H_i$ and $\bigcap_{i=1}^{n} H_i$ are also members of I.

Example: Let G be a non-trivial graph and H be an induced subgraph of G. Then $I = \{G, \phi, H, H^c\}$ is a σ-field containing H.

Example: Let G be a non-trivial graph and I be the collection of all induced subgraphs of G together with the empty graph ϕ. Then I is a σ-field.

Definition 2.6: A *graph function* μ_v defined on I is said to be finite if $\mu_v(H) < \infty$ for every induced subgraph $H \in I$. ■

Definition 2.7: Let G be a graph and I be the σ-field of G and let μ_v be an extended real valued graph function defined on I. We say that μ_v is *finitely additive* if

$$\mu_v\left(\biguplus_{i=1}^{n} H_i \right) = \sum_{i=1}^{n} \mu_v(H_i)$$

for all vertex disjoint induced subgraphs $H_i \in I$, $i = 1, 2, \ldots, n$. ■

Definition 2.8: Let I be a σ-field of induced subgraphs of G and μ_v be an extended real valued graph function defined on I. We say that μ_v is *countably additive* on I if

$$\mu_v\left(\bigvee_{i=1}^{\infty} H_i\right) = \sum_{i=1}^{\infty} \mu_v(H_i)$$

for all vertex disjoint induced subgraphs $H_i \in I$, $i = 1, 2, \ldots, n$. ■

Definition 2.9: A finitely additive graph function μ_v on the field I is said to be σ-finite if $G = \bigvee_{i=1}^{\infty} H_i$ for $H_i \in I$ and $\mu_v(H_i) < \infty$. ■

Theorem 2.10: Let μ_v be a non-negative finitely additive graph function on the field I. Then

(1) $\mu_v(\phi) = 0$.

(2) If $H_1, H_2 \in I$ and $H_2 \subset H_1$, then $\mu_v(H_2) \le \mu_v(H_1)$.

(3) $\mu_v(H_1 \vee H_2) + \mu_v(H_1 \cap H_2) = \mu_v(H_1) + \mu_v(H_2)$ for all $H_1, H_2 \in I$.

(4) If μ_v is non-negative, then for all $H_i \in I$

$$\mu_v\left(\bigvee_{i=1}^{n} H_i\right) \le \sum_{i=1}^{n} \mu_v(H_i).$$

Proof:

(1) Let $H \in I$ be such that $\mu_v(H) < \infty$. Then

$$\mu_v(H) = \mu_v(H \vee \phi) = \mu_v(H) + \mu_v(\phi) \Rightarrow \mu_v(\phi) = 0.$$

(2) Let $H_1, H_2 \in I$ and let $H_2 \subset H_1$. Then

$$H_1 = H_2 \vee (H_1 - H_2).$$

Therefore, $\mu_v(H_1) = \mu_v(H_2) + \mu_v(H_1 - H_2)$, which implies that $\mu_v(H_2) = \mu_v(H_1) - \mu_v(H_1 - H_2)$. Since $\mu_v(H_1 - H_2) \ge 0$, then $\mu_v(H_2) \le \mu_v(H_1)$.

(3) We know that $H_1 = (H_1 \cap H_2) \vee (H_1 \cap H_2^c)$, therefore

(4) $$\mu_v(H_1) = \mu_v(H_1 \cap H_2) + \mu_v(H_1 \cap H_2^c)$$

(5) $$\mu_v(H_2) = \mu_v(H_1 \cap H_2) + \mu_v(H_1^c \cap H_2).$$

From **(4)** and **(5)** it follows that

$$\begin{aligned}\mu_v(H_1) + \mu_v(H_2) &= \mu_v(H_1 \cap H_2) + \{\mu_v(H_1 \cap H_2) + \mu_v(H_1 \cap H_2^c) + \mu_v(H_1^c \cap H_2)\} \\ &= \mu_v\Big((H_1 \cap H_2) \vee (H_1 \cap H_2^c) \vee (H^c_{1_{v_G}} \cap H_2)\Big) \\ &= \mu_v(H_1 \cap H_2) + \mu_v(H_1 \vee H_2).\end{aligned}$$

Hence,

$$\mu_v(H_1 \vee H_2) = \mu_v(H_1) + \mu_v(H_2) - \mu_v(H_1 \cap H_2).$$

(4) We prove that

$$\mu_v\left(\bigvee_{i=1}^{n} H_i\right) \le \sum_{i=1}^{n} \mu_v(H_i),$$

by mathematical induction.

By (3), $\mu_v(H_1 \vee H_2) \le \mu_v(H_1) + \mu_v(H_2)$; that is,

$$\mu_v\left(\bigcup_{i=1}^{2} H_i\right) \le \sum_{i=1}^{2} \mu_v(H_i).$$

Assume that the result is true for all values of $i < n$; that is,

$$\mu_v\left(\bigcup_{i=1}^{n-1} H_i\right) \le \sum_{i=1}^{n-1} \mu_v(H_i).$$

Now consider

$$\begin{aligned}
\mu_v\left(\bigcup_{i=1}^{n} H_i\right) &= \mu_v\left(\left(\bigcup_{i=1}^{n-1} H_i\right) \cup H_n\right) \\
&\le \mu_v\left(\bigcup_{i=1}^{n-1} H_i\right) + \mu_v(H_n) \\
&\le \sum_{i=1}^{n-1} \mu_v(H_i) + \mu_v(H_n) \\
&\le \sum_{i=1}^{n} \mu_v(H_i).
\end{aligned}$$

■

Definition 2.11: A non-negative extended real valued countably additive graph function μ_v on *I* (that is, there exists at least one *H* such that $\mu_v(H) < \infty$) is called a *vertex measure* on *I*. The members of *I* are said to be *vertex measurable graphs*. ■

Definition 2.12: A *vertex measure space* is an ordered triple (G, I, μ_v) where *G* is a graph, *I* is a σ-field in *G*, and μ_v is a vertex measure on *I*. ■

Example: Let *G* be a graph and *I* be a σ-field of all induced subgraphs of *G*, together with the null graph ϕ. Define $\mu_v(H)$ to be the number of vertices in *H*. If $H \in I$ and *H* has *n* vertices, $n = 0, 1, 2, \dots$, then $\mu_v(H) = n$. This vertex measure, in particular, is called the *vertex counting measure* on I.

3. Main Results

Theorem 3.1: Let μ_v be a vertex measure, not identically ∞ on a σ-field *I*. Then

(1) $\mu_v(\phi) = 0$.

(2) If $H_1, H_2 \in I$ and $H_2 \subset H_1$, then $\mu_v(H_2) \le \mu_v(H_1)$.

(3) $\mu_v(H_1 \cup H_2) + \mu_v(H_1 \cap H_2) = \mu_v(H_1) + \mu_v(H_2)$ for all $H_1, H_2 \in I$.

(4) $\mu_v\left(\bigcup_{i=1}^{\infty} H_i\right) \le \sum_{i=1}^{\infty} \mu_v(H_i)$ for $H_i \in I$.

Proof: Statements (1), (2), and (3) follow directly from Theorem 2.10.

Statement (4) of Theorem 2.10 asserts that

$$\mu_v\left(\bigcup_{i=1}^{n} H_i\right) \le \sum_{i=1}^{n} \mu_v(H_i).$$

Taking limits as $n \to \infty$ on both sides

$$\lim_{n \to \infty}\left(\bigcup_{i=1}^{n} H_i\right) \le \lim_{n \to \infty} \sum_{i=1}^{n} \mu_v(H_i),$$

that is,

$$\mu_v\left(\bigvee_{i=1}^{\infty} H_i\right) \le \sum_{i=1}^{\infty} \mu_v(H_i). \qquad \blacksquare$$

Theorem 3.2: Let $\{H_n\}$ be a sequence of induced subgraphs of *G*, such that $H_n \subset H_{n+1}$ for all $n \ge 1$ and $\left(\bigvee_{n \ge 1} H_n\right) \in I$. Then

$$\mu_v\left(\bigvee_{n \ge 1} H_n\right) = \lim_{n \to \infty} \mu_v(H_n).$$

Proof: If $\mu_v(H_n) = \infty$ for some $n = n_0$, then $\mu_v(H_n) = \infty$ for all $n \ge n_0$ and $\mu_v\left(\bigvee_{n \ge 1} H_n\right) = \infty$. Hence the result is true.

Suppose that $\mu_v(H_n) < \infty$ for all $n \ge 1$. Set $L_n = H_n - H_{n-1}$ for all $n \ge 1$, with $H_0 = \phi$. Then $\mu_v(L_n) = \mu_v(H_n - H_{n-1}) = \mu_v(H_n) - \mu_v(H_{n-1})$ for $n \ge 1$.

Since $\{L_n\}$ is a disjoint collection of induced subgraphs in *I* with

$$\bigvee_{n \ge 1} L_n = \bigvee_{n \ge 1} H_n,$$

then

$$\begin{aligned}
\mu_v\left(\bigvee_{n \ge 1} H_n\right) &= \mu_v\left(\bigvee_{n \ge 1} L_n\right) \\
&= \sum_{i=1}^{\infty} \mu_v(L_n). \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \{\mu_v(H_n) - \mu_v(H_{n-1})\} \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \mu_v(H_N).
\end{aligned}$$

Hence,

$$\mu_v\left(\bigvee_{n \ge 1} H_n\right) = \lim_{n \to \infty} \mu_v(H_n). \qquad \blacksquare$$

Theorem 3.3: Let $\{H_n\}$ be a sequence of induced subgraphs in *I* such that $H_{n+1} \subset H_n$ for all $n \ge 1$ and $H = \bigcap_{n \ge 1} H_n \in I$. Also, let $\mu_v(H_{n_0}) < \infty$ for some $n_0 \in N$, then $\lim_{n \to \infty} \mu_v(H_n) = \mu_v(H)$.

Proof: Without loss of generality we may assume that $n_0 = 1$, so that $\mu_v(H_1) < \infty$. Let $F_n = H_1 - H_n$ for $n \ge 1$ and $F_\infty = H_1 - H$. Then F_n and F_∞ both belong to *I* and $F_n \to F_\infty$. Hence, by Theorem 3.2, $\mu_v(F_n) \to \mu_v(F_\infty)$; that is, $\mu_v(H_1 - H_n) \to \mu_v(H_1 - H)$. This implies, $\mu_v(H_1) - \mu_v(H_n) \to \mu_v(H_1) - \mu_v(H)$, which implies that $\mu_v(H_n) \to \mu_v(H)$. ■

Definition 3.4: A vertex measure $\mu_v: I \to \mathbb{R}$ is called a *vertex probability measure* if $\mu_v(G) = 1$. ■

Note: Generally, a vertex probability measure μ_v is denoted by P_v.

Definition 3.5: A *vertex probability measure space* is an ordered triple (G, I, P_v). ■

Example: Let *G* be a finite graph and let *I* be a σ-field of induced subgraphs of *G*. Define a graph function $P_v: I \to \mathbb{R}$ by

$$P_v(H) = \frac{p(H)}{p(G)} \text{ for all } H \in I.$$

Then $P_v(\phi) = 0$, $0 \le P_v(H) \le 1$ for all $H \in I$, and $P_v(G) = 1$. Hence P_v is a vertex probability measure defined on *I*.

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ON AN OPEN PROBLEM OF R. BALAKRISHNAN AND THE ENERGY OF PRODUCTS OF GRAPHS

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Abstract

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of G. If G is an r-regular graph on n vertices then $E(G) \le r + \sqrt{r(n-1)(n-r)} = B_2$ and this bound is sharp. Recently Balakrishnan showed that for $\varepsilon > 0$, there exist infinitely many r-regular graphs G such that $E(G)/B_2 < \varepsilon$ and then asked: Does there exist an r-regular graph G on n vertices for some r, $0 < r < n$, such that $E(G)/B_2 < 1 - \varepsilon$. In this paper we show that this problem has a solution for all $n \ge 5$ where $n \equiv 1 \pmod 4$. The energy of a tensor product and join of graphs is discussed.

1. Introduction

Let G be a simple undirected graph on n vertices and m edges. Let $A(G)$ be the *adjacency matrix* of G. The *characteristic polynomial* of G is defined by $\Phi(G;\lambda) = \det(\lambda I - A(G))$, where I is an identity matrix. The roots of the equation $\Phi(G;\lambda) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the *eigenvalues* of G **[1]**. The *energy of a graph G* is defined by $E(G) = \sum_{i=1}^{n} |\lambda_i|$ (see **[2]**).

The complete graph K_n has energy $2(n-1)$. If $E(G) > 2(n-1)$, then G is called *hyperenergetic*, otherwise it is called *non-hyperenergetic* **[3]**–**[10]**.

2. An Open Problem of R. Balakrishnan

For a graph G on n vertices and m edges it is shown **[7]** that

(1) $$E(G) \le \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]} = B_1.$$

However, if G is an r-regular graph then **[10]**

(2) $$E(G) \le r + \sqrt{r(n-1)(n-r)} = B_2.$$

There are regular graphs for which the bound B_2 is attained. For example, $E(K_n) = B_2$ and the complement of K_n, $\overline{K}_n$, satisfies $E(\overline{K}_n) = 0$.

Recently R. Balakrishnan **[11]** has shown that for $\varepsilon > 0$, there exist infinitely many r–regular graphs G such that $E(G)/B_2 < \varepsilon$ and he posed the following problem.

> **Problem [11]:** Given a positive integer $n \ge 3$, does there exist an r-regular graph G for some r, $0 < r < n$, such that $E(G)/B_2 < 1 - \varepsilon$, where $B_2 = r + \sqrt{r(n-1)(n-r)}$? □

Because $\varepsilon > 0$, $1 - \varepsilon < 1$. Hence, graphs whose energy is almost the same as B_2 need to be investigated.

Here we give an affirmative answer to this problem for all $n \ge 5$, $n \equiv 1 \pmod 4$.

A regular graph G on n vertices that is neither complete nor empty is called a *strongly regular graph* **[12]** with parameters (n, r, a, c) if it is r-regular, every pair of adjacent vertices has a common neighbors, and every pair of distinct nonadjacent vertices has c common neighbors. A simple example is the cycle C_5, which is a strongly regular graph with parameters $(5, 2, 0, 1)$.

A connected regular graph G is strongly regular if and only if it has exactly three distinct eigenvalues **[13]**.

The *eigenvalues* of a strongly regular graph with parameters (n, r, a, c) are:

$$r, \ \frac{(a-c)+\sqrt{(a-c)^2+4(r-c)}}{2}, \ \text{and} \ \frac{(a-c)-\sqrt{(a-c)^2+4(r-c)}}{2},$$

with multiplicities

$$1, \ \frac{1}{2}\left[(n-1)-\frac{2r+(n-1)(a-c)}{\sqrt{(a-c)^2+4(r-c)}}\right], \ \text{and} \ \frac{1}{2}\left[(n-1)+\frac{2r+(n-1)(a-c)}{\sqrt{(a-c)^2+4(r-c)}}\right], \ \text{respectively.}$$

The strongly regular graph with parameters $\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right)$ is called a *Paley graph* **[14]**.

Theorem 1: Given any positive integer $n \geq 5$, $n \equiv 1 \pmod 4$, and $\varepsilon > 0$, there exists an r-regular graph G on n vertices for some r, $0 < r < n$, such that $E(G)/B_2 > 1 - \varepsilon$.

Proof: Let G be the Paley graph on n vertices. Then G is a strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$. G is a is regular graph of degree $(n-1)/2$ and its eigenvalues are $(n-1)/2$, $(-1+\sqrt{n})/2$, and $(-1-\sqrt{n})/2$ with multiplicities 1, $(n-1)/2$, and $(n-1)/2$, respectively. Therefore,

$$E(G) = \left|\frac{n-1}{2}\right| + \left|\frac{-1+\sqrt{n}}{2}\right|\left(\frac{n-1}{2}\right) + \left|\frac{-1-\sqrt{n}}{2}\right|\left(\frac{n-1}{2}\right) = \frac{(n-1)(2\sqrt{n}+1)}{4}.$$

From Equation **(2)**, the bound B_2 on $E(G)$ is

$$B_2 = \frac{n-1}{2} + \sqrt{\left(\frac{n-1}{2}\right)(n-1)\left(n-\frac{n-1}{2}\right)} = \frac{(n-1)(1+\sqrt{n+1})}{2}.$$

Hence,

$$\frac{E(G)}{B_2} = \frac{2\sqrt{n}+1}{2(1+\sqrt{n+1})} \ \text{tends to 1 as} \ n \to \infty.$$

This shows that there exists an r-regular graph G on n vertices, $n \equiv 1 \pmod 4$ for some r, $0 < r < n$, such that $E(G)/B_2 > 1 - \varepsilon$. ■

Table: Values of $E(G)$, B_2, and $E(G)/B_2$

n	$E(G) = \frac{(n-1)(2\sqrt{n}+1)}{4}$	$B_2 = \frac{(n-1)(1+\sqrt{n+1})}{2}$	$\frac{E(G)}{B_2}$
101	527.4937811	554.9752469	0.950481
525065	190365547.31101018	190496994.46400146	0.999309977
1011101	508601085.7	508854112.1	0.999502752
102496524	518865929169.129646	518891555830.893789	0.999950612

Thus, the problem of Balakrishnan has a partial solution in the form of Paley graphs on $n \geq 5$ vertices, $n \equiv 1 \pmod 4$.

3. The Energy of a Tensor Product and the Join of Graphs

Definition: The *tensor product* of two graphs G_1 and G_2 is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. ■

Theorem 2 [11]: If G_1 and G_2 are any two graphs, then $E(G_1 \otimes G_2) = E(G_1)E(G_2)$. ■

Theorem 3: If G_1 and G_2 are hyperenergetic graphs with n_1 and n_2 vertices, respectively, then $G_1 \otimes G_2$ is hyperenergetic.

Proof: From the table given in **[1]**, it is easy to see that there is no hyperenergetic graph for $n \le 5$. Hence, we consider $n_1, n_2 > 5$.

The graphs G_1 and G_2 are hyperenergetic. Therefore, $E(G_i) > 2(n_i - 1)$, $i = 1, 2$. The order of $G_1 \otimes G_2$ is $n_1 n_2$ and $E(G_1 \otimes G_2) = E(G_1)E(G_2) > 2(n_1 - 1)2(n_2 - 1)$.

$G_1 \otimes G_2$ is hyperenergetic if $2(n_1 - 1)2(n_2 - 1) > 2(n_1 n_2 - 1)$. This gives either

$$n_1 > \frac{2n_2 - 3}{n_2 - 2} = 2 + \frac{1}{n_2 - 2} \quad \text{or} \quad n_2 > \frac{2n_1 - 3}{n_1 - 2} = 2 + \frac{1}{n_1 - 2}.$$

Both inequalities are true for all $n_1, n_2 > 5$. Hence the result. ■

Corollary 4: Let G_i be a hyperenergetic graph with n_i vertices, $i = 1, 2, \ldots, k$, $k \ge 2$. Then $G_1 \otimes G_2 \otimes \ldots \otimes G_k$ is hyperenergetic. ■

Theorem 5: If G is a non-hyperenergetic graph with n vertices, then $K_2 \otimes G$, $C_4 \otimes G$, and $\bar{C}_4 \otimes G$ are non-hyperenergetic.

Proof: G is non-hyperenergetic. Therefore, $E(G) \le 2n - 2$. The order of $K_2 \otimes G$ is $2n$ and $E(K_2) = 2$. Therefore, $E(K_2 \otimes G) = 2E(G) \le 2(2n - 2) = 2(2n - 1) - 2 < 2(2n - 1)$. Similarly, one can show that $C_4 \otimes G$ and $\bar{C}_4 \otimes G$ are non-hyperenergetic. ■

Definition: The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is the graph obtained by joining every vertex of G_1 with every vertex of G_2. ■

Theorem 6 [15]: If G_i is a regular graph of degree r_i with n_i vertices, $i = 1, 2$, then

(3)
$$\Phi(G_1 + G_2;\lambda) = \frac{\Phi(G_1;\lambda)\Phi(G_2;\lambda)}{(\lambda - r_1)(\lambda - r_2)}[(\lambda - r_1)(\lambda - r_2) - n_1 n_2]. \quad ■$$

Theorem 7: If G_i is a regular graph of degree r_i with n_i vertices, $i = 1, 2$, then

$$E(G_1 + G_2) = E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2).$$

Proof: From **(3)**, we obtain

$$\Phi(G_1 + G_2;\lambda)(\lambda - r_1)(\lambda - r_2) = \Phi(G_1;\lambda)\Phi(G_2;\lambda)[(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

Let $P_1 = \Phi(G_1 + G_2;\lambda)(\lambda - r_1)(\lambda - r_2)$ and $P_2 = \Phi(G_1;\lambda)\Phi(G_2;\lambda)[(\lambda - r_1)(\lambda - r_2) - n_1 n_2]$. The roots of $P_1 = 0$ are the eigenvalues of $G_1 + G_2$ and r_1, r_2. Therefore, the sum of the absolute values of the roots of $P_1 = 0$ is

(4)
$$E(G_1 + G_2) + r_1 + r_2.$$

The roots of $P_2 = 0$ are the eigenvalues of G_1 and G_2, and

$$\frac{r_1 + r_2 \pm \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}}{2}.$$

Therefore, the sum of the absolute values of the roots of $P_2 = 0$ is

$$E(G_1) + E(G_2) + \left|\frac{r_1 + r_2 + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}}{2}\right| + \left|\frac{r_1 + r_2 - \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}}{2}\right|.$$

This is equal to

(5)
$$E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)}.$$

Since $P_1 = P_2$, equating **(4)** and **(5)** we obtain

$$E(G_1 + G_2) = E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2). \qquad ■$$

If $r_1 = r_2$ and $n_1 = n_2$, then from Theorem 7, we have following corollary.

Corollary 8: If G_1 and G_2 are regular graphs of degree r on n vertices, then

$$E(G_1 + G_2) = E(G_1) + E(G_2) + 2n - 2r. \qquad ■$$

Theorem 9: Let G_i be a regular graph of degree r_i with n_i vertices, $i = 1, 2$. If G_1 and G_2 are hyperenergetic, then $G_1 + G_2$ is also hyperenergetic.

Proof: Let G_i be a hyperenergetic graph. Therefore, $E(G_i) > 2(n_i - 1)$, $i = 1, 2$.

$$\begin{aligned} E(G_1 + G_2) &= E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2) \\ &> 2(n_1 - 1) + 2(n_2 - 1) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2). \end{aligned}$$

The order of $G_1 + G_2$ is $n_1 + n_2$. To show that $G_1 + G_2$ is hyperenergetic, it is sufficient to show that

(6) $$2(n_1 - 1) + 2(n_2 - 1) + \sqrt{(r_1 + r_2)^2 + 4(n_1 n_2 - r_1 r_2)} - (r_1 + r_2) > 2(n_1 + n_2 - 1).$$

Simplification of **(6)** leads to $n_1 n_2 > (r_1 + 1)(r_2 + 1)$, which is true since $r_1 < n_1 - 1$ and $r_2 < n_2 - 1$. Hence the result. ■

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[GTN LV:7] STRONGLY *k*-INDEXABLE UNICYCLIC GRAPHS

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Abstract

Given any positive integer k, Acharya and Hegde called a (p, q)-graph, $G = (V, E)$, strongly k-indexable if there exists a bijection $f\colon V \to \{0, 1, \ldots, p-1\}$ such that $f^{+}(E(G)) = \{k, k+1, \ldots, k+q-1\}$, where $f^{+}(uv) = f(u) + f(v)$ for any edge $uv \in E(G)$. In particular, they called G strongly indexable when $k = 1$. Kotzig and Rosa called a (p, q)-graph, $G = (V, E)$, edge-magic if it admits an edge-magic labelling of G that is defined as a bijection $f\colon V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that there exists a constant s (called the magic number of f) with $f(u) + f(v) + f(uv) = s$, $\forall uv \in E(G)$. Enomoto *et al.* called an edge-magic labelling f of G super-edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$ and $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$ and called G super-edge-magic if there exists a super-edge-magic labelling of G. Figueroa-Centeno *et al.* established that strongly k-indexable labelling extends to super-edge magic labelling. In this paper we prove that for $k = 1$ the class of strongly k-indexable graphs is a proper subclass of super-edge-magic graphs and prove that any graph can be embedded in a strongly indexable graph. Furthermore, we give an algorithmic characterization of strongly indexable unicyclic graphs, thereby arriving at a partial solution to the open problem of characterizing graphs in the class U_n, $n \geq 3$, that are super-edge-magic.

1. Introduction

By a graph we mean a finite, undirected, connected graph with no loop or multiple edge. Terms not defined here are used in the sense of Harary **[1]**.

As a special case of *arithmetic graphs*, Acharya and Hegde **[2][3]**, introduced the concept of an *indexable graph* as follows: Let $G = (V, E)$ be a (p, q)-graph. A *labelling* of G is a bijection $f\colon V \to \{0, 1, \ldots, p-1\}$. A labelling f is called an *indexer* if f is such that the induced edge function $f^{+}\colon E(G) \to \mathbb{N}$, from $E(G)$ into the set $\mathbb{N}$ of natural numbers, defined by the rule: $f^{+}(uv) = f(u) + f(v)$, $\forall uv \in E(G)$, is injective. In particular, if

$$f^{+}(E(G)) := \{f^{+}(uv) : uv \in E(G)\} = \{k, k+1, \ldots, k+q-1\}$$

for some positive integer k then f is called a *k-strong indexer* of G. A graph G is said to be *indexable* (*k-strongly indexable*) if it admits an indexer (k-strong indexer). In particular, if $k = 1$ in this definition, then f is called simply a *strong indexer* of G. The graph G is said to be *strongly indexable* if it admits a strong indexer.

The following result, obtained by Arumugam and Germina **[3]**, settled a conjecture by Acharya and Hegde **[2]**.

Theorem 1.1 [3]: Every connected graph with at most one cycle is indexable. ■

Corollary 1.2 [3]: All unicyclic graphs are indexable. ■

Arumugam and Germina also constructed several infinite classes of strongly indexable graphs. In this paper, we characterize strongly indexable unicyclic graphs, settling an open problem stated by Acharya and Hegde **[2]**, and prove that any graph can be embedded as an induced subgraph in a strongly indexable graph. For any given positive integer d, Acharya and Hegde **[2]** constructed strongly indexable graphs of diameter d in U_3, the class of *unicyclic graphs* in each of which the unique cycle is a triangle, and posed the open problem to determine, in general, which graphs in U_3 are strongly indexable. We need the following previously known results.

Theorem 1.3 [2]: Every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. ■

Corollary 1.4 [2]: If G is a strongly indexable graph with a triangle as a component then any strong indexer of G must assign zero to a vertex of the triangle in G. ■

Theorem 1.5 [3]: Let G be a unicyclic graph with the unique triangle (u, v, w, u), such that each vertex different from u, v, and w has degree one. Let m be the number of pendant vertices adjacent to u. Then, G is strongly indexable if and only if there exist distinct positive integers x and y such that one of the following holds:
(1) $m_u = x + y - 3 + m_v(x-1) + m_w(y-1)$,
(2) $m_v = x + y - 3 + m_u(x-1) + m_w(y-1)$, or
(3) $m_w = x + y - 3 + m_v(x-1) + m_u(y-1)$. ■

Theorem 1.6 [3]: Let G be a unicyclic graph consisting of the unique triangle (v_1, v_2, v_3, v_1), with $\deg(v_2) = \deg(v_3) = 2$, a path $P = (v_1, u_1, u_2, \ldots, u_n)$ of length n and k pendant edges adjacent to v_1. Then G is strongly indexable if and only if $k = a_n$, where $a_2 = x + y + z - 5$, $a_3 = 2x + 2y + z - 6$, $a_n = a_{n-1} + a_{n-2} + n$, for $n > 3$, and x, y, z are distinct positive integers with $z \neq x + y$. ■

In view of Theorems 1.5 and 1.6, we need to settle the case when, in the triangle of the given unicyclic graph, the vertex assigned zero has degree greater than two.

2. Main Results

The *breadth first search* (BFS) algorithm provides a systematic way to visit all the vertices of a connected graph G starting from a vertex v in G, marking v as visited, and then the unvisited vertices adjacent to v are visited next. The vertex v is called the *father* of the subsequent vertices, which are called the *children* of v. Unvisited vertices adjacent to the children of v are visited next, and so on. In this process, the edges of G are partitioned into two subsets T and B, where T is the set of edges traversed during the search and B is the set of remaining edges. The edges in T form a *spanning tree* of G, and hence, are called *tree edges* and the edges of B are called *back edges* (see Golumbic **[4]**).

Theorem 2.1: Let G be a connected unicyclic graph of order p, with a unique triangle described by (u_1, u_2, u_3, u_1). Then a bijective function $f\colon V \to \{0, 1, \ldots, p-1\}$ is a strong indexer of G if and only if there exists a BFS algorithm starting at the vertex labelled zero, such that the following conditions are satisfied.

(1) The vertex labelled zero is one of the vertices of the triangle, say u_1.

(2) There exist distinct positive integers $z_1, z_2, \ldots, z_{m_1}$ such that $f(u_{1j}) = z_j$, $1 \le j \le m_1$, where $N(u_1) = \{u_{11}, u_{12}, \ldots, u_{1m_1}\}$ and no two z_is have a sum equal to z_j for any j, $1 \le j \le m_1$.

(3) There exist pendant vertices adjacent to u_1, with $a_1 = z_{m_1} - m_1$ and $a_n = a_{n-1} + f(F(u_{n-1})) - 1$ for $n \ge m_1 + 1$, where $F(u_n)$ is the father vertex of u_n in the BFS algorithm.

Proof:

Necessity: Let f be a strong indexer of G. Let $d(u_i) > 2$. Apply the BFS algorithm, starting from vertex x, labelled zero. Then by Corollary 1.4, $x = u_1$. Let $f(u_{1j}) = z_j$, $1 \le j \le m_1$, with $f(u_2) + f(u_3) \neq z_j$ for any j, $1 \le j \le m_1$. Since f is a strong indexer of G, all integers other than $z_1, z_2, \ldots, z_{m_1}$ should be labelled among the neighborhood vertices (i.e., children) of u_1. Without loss of generality, assume $z_1 < z_2 < \ldots < z_{m_1}$. All integers $1, 2, \ldots, z_1 - 1, z_2 + 1, z_2 + 2, \ldots, z_3 - 1, z_3 + 1, z_3 + 2, \ldots, z_{m_1} - 1$ must appear as labels among the children of u_1. There are precisely $z_{m_1} - m_1$ such numbers, and these must have been assigned to the children of u_1, and hence, require $z_{m_1} - m_1$ pendant vertices adjacent to u_1. However, $f(u_2) + f(u_3)$ must appear as a label of a vertex in the neigborhoods of the neighborhood of u_1, say u_{11}. Now, all integers other than the already labelled integers and less than $f(u_{11}) + f(u_2) + f(u_3)$ must have been assigned to the children of u_1, and this number is precisely

$$z_{m_1} - m_1 + [f(u_{11} - 1)] = a_1 + f(F(u_1)) - 1.$$

Now, assume that the result is true for a_{n-1}, where $a_{n-1} = a_{n-2} + f(F(u_{n-2})) - 1$. Since *f* is a strong indexer of *G*, all integers less than $f(u_{n-1}) + f(u_{n-2})$, other than those already used, must have been assigned to the children of u_1 and the number of such integers is precisely $a_{n-1} + f(F(u_{n-1})) - 1$. That is, there must be $a_n = a_{n-1} + f(F(u_{n-1})) - 1$ pendant vertices adjacent to u_1. Hence, by the principle of induction, there exist $a_n = a_{n-1} + f(F(u_{n-1})) - 1$ pendant vertices adjacent to u_1.

<u>Conversely</u>: Let *G* be a (*p*, *q*)-unicyclic graph with the unique triangle (u_1, u_2, u_3, u_1). Let the vertices of *G* be visited, using the BFS algorithm, starting from the vertex u_i, where $\deg(u_1) > 2$ for $i = 1, 2, 3$. Let u_n denote the *n*th vertex visited in the BFS algorithm. Let $\deg(u_i) = m_i$ for $i = 1, 2, 3$. Assume that there exist a_n pendant vertices adjacent to u_1, where $a_n = a_{n-1} + f(F(u_{n-1})) - 1$ with $a_1 = z_{m_1} - m_1$. Define $f: V \to \{0, 1, \ldots, p-1\}$ as follows: $f(u_1) = 0$. Visit the vertices in the neighborhood of u_1 and assign the values $z_1, z_2, \ldots, z_{m_1}$, with $f(u_2) + f(u_3) \neq z_j$ for any *j*, $1 \le j \le m_1$, in the order in which they are visited.

Label the $a_1 = z_{m_1} - m_1$ pendant vertices in the neighborhood of u_1 with the unused integers less than z_{m_1}. Define $f(u_{m_1}) + 1 = f(u_2) + f(u_3)$, where u_{m_1} is the vertex in the neighborhood of u_2. Continue the process of visiting the vertices using the BFS algorithm and label the vertices in the order in which they are visited by $f(u_i) = f(u_{i-1}) + f(u_{i-2})$, for each $i > m_1 + 1$ and the a_i pendant vertices with the unlabeled distinct integers less than $f(u_{i-1}) + f(u_{i-2})$. Then $f(V(G)) = \{0, 1, \ldots, p-1\}$ and $f^+(E(G)) = \{1, 2, \ldots, q\}$. Hence, $f: V \to \{0, 1, \ldots, p-1\}$ is a strong indexer of *G*. ■

Notice, from the proof of the above theorem, that the strong indexer generated by the algorithm contains the first *n* terms of a *Fibonacci sequence* with zero as its first term; it also brings out the following converse part of this statement.

Corollary 2.2: For any integer $n \ge 3$, given the set $A_{n+1}(\mathbb{F})$ consisting of the first $n + 1$ terms of a Fibonacci sequence $\mathbb{F}$ with zero as its first term, there exists a graph in U_3 together with a strong indexer that uses all the elements of $A_{n+1}(\mathbb{F})$. ■

Remark 2.3: The graph in U_3 having a strong indexer *f* such that $A_n(\mathbb{F}) \subseteq f(G)$ may not be unique.

This raises an interesting new problem.

Problem 2.4: Given the set $A_{n+1}(\mathbb{F})$ of the first $n + 1$ terms of a Fibonacci sequence $\mathbb{F}$ with zero as its first term, determine the class of all minimal nonisomorphic strongly indexable graphs in U_3 for which $A_n(\mathbb{F}) \subseteq f(G)$ for some strong indexer *f*. □

Our next result shows that there is no *forbidden subgraph characterization* for strongly indexable graphs.

Theorem 2.5: Every graph can be embedded as an induced subgraph in a strongly indexable graph.

Proof: If the given graph is strongly indexable then there is nothing to show. Hence, let *G* be a graph that is not strongly indexable. There are two cases.

<u>Case 1</u>: Let the graph *G* have at least one triangle as a subgraph. Let (x_1, x_2, x_3, x_1) be one such triangle. Choose one of the vertices of this triangle, say $a \in V(G)$, with $\deg(a) > 2$. Starting from *a*, visit the vertices of the graph using the BFS algorithm, visiting vertices not on the triangle first. We define $f: V(G) \to \mathbb{N}$ as follows: Let $f(a) = 0$. The *i*th vertex visited, u_i, in the BFS algorithm is labelled so that $f(u_i) = f(u_{i-1}) + f(u_{i-2})$. If *G* has cycles other than the cycle (x_1, x_2, x_3, x_1), as we visit vertices on these cycles there occurs a back edge at each cycle and hence we obtain two edge values simultaneously. In this case, the next vertex visited should be assigned the maximum of the two edge values and the minimum value assigned to a new isolated vertex. Now, assign all the remaining unlabeled integers less than $f(u_{i-1}) + f(u_i)$

to vertices in the neighborhood of a so as to have all consecutive integers from $\{1, 2, \ldots, f(u_{i-1}) + f(u_{i-2})\}$ as edge labels. Thus, add as many isolated vertices as there are back edges produced at this stage by the BFS algorithm and $f(u_i)$ minus the number of tree edges encountered pendant edges adjacent to vertex a. Since G is a finite (p, q)-graph, the algorithm terminates when the pth vertex is visited by the BFS algorithm, producing a graph H that is strongly indexed by f and contains G as an induced subgraph.

Case 2: Let G be the given graph, which is triangle-free. Take a new vertex, x, and join it to any two vertices a and b of G, so that (x, a, b, x) is a triangle. Then, proceed as in Case 1 to complete the proof. ■

Remark 2.6: Notice that the proof of Theorem 2.5 remains valid even if the given graph G is locally finite and countably infinite.

3. Connection to Super-Edge-Magic Graphs

In the previous section, we completely characterized strongly indexable unicyclic graphs. They turn out to be a specific subclass of the class U_3 of all connected unicyclic graphs in which the length of the unique cycle is three. Denote by U_n the class of all connected unicyclic graphs in each of which the length of the unique cycle is precisely n. If $n > 3$, then by Theorem 1.3 it follows that no graph in U_n is strongly indexable. If in any graph $G \in U_n$, $n > 3$, we join two vertices u and v that are at distance two by a new edge uv, we obtain a graph $H_{uv} = G + uv$ containing the triangle (u, x, v, u), where x is the vertex on a geodesic (u, x, v) in G. Now, some such graphs H of the form $H_{uv} := G + uv$, $uv \in E(G)$ could be strongly indexable. It would be interesting to characterize such strongly indexable graphs since they form a subclass of outerplanar graphs with cyclomatic number two.

Kotzig and Rosa **[5]** called a (p, q)-graph $G = (V, E)$ *edge magic* if it admits an *edge-magic labelling* of G, defined as a bijection $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that there exists a constant, s (called the *magic number* of f), with $f(u) + f(v) + f(uv) = s$, $\forall uv \in E(G)$. Enomoto *et al.* **[6]** called an edge-magic labelling f of G *super-edge-magic* if $f(V(G)) = \{1, 2, \ldots, p\}$ and $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$, and called G *super-edge-magic* if there exists a super-edge-magic labelling of G. The following Lemma of Figuero-Centro *et al.* **[7]** provides an interesting connection between k-strongly indexable graphs and super-edge-magic graphs.

> **Lemma 3.1:** A (p, q)-graph G is super-edge-magic if and only if there exists a bijective function $f: V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f can be extended to a super-edge-magic labeling of G with constant $c = p + q + s$, where
>
> $$s = \min\{c - (p+1), c - (p+2), \ldots, c - (p+q)\}.$$ ■

The following remark is immediate from Lemma 3.1.

Remark 3.2: For any positive integer k, if a (p, q)-graph G is strongly k-indexable, then every strong k-indexer of G extends G to a super-edge-magic graph with magic number equal to $p + q + k + 2$. Conversely, if G is super-edge-magic with magic number $c \geq 4$, then G has a strong k-indexer with $k = c - 2 - p - q$.

Remark 3.3: Let f be a super-edge-magic labelling of a (p, q)-graph G and let $g(u) = f(u) - 1$, $\forall u \in V(G)$, so that g is a strong k-indexer of G with $k = c - 2 - p - q$. If c is the magic number of the super-edge-magic labeling of G, then $c \geq 1 + 2 + p + 1 = p + 4$ and $c = 6$ results in an isolated edge, which is the trivial case. Hence, $c \geq 6$. Also, $\min\{g(v) : v \in V(G)\} = 0$ and $\min\{g(e) : e \in E(G)\} = k$. Hence,

$$k = c - 2 - (p+q) \geq 6 - 2 - k \Leftrightarrow 2k \geq 4 \Leftrightarrow k \geq 2.$$

From Remark 3.3, we see that the converse of Remark 3.2 is likely to fail for $k = 1$ and, in fact, it does in view of Theorem 1.3 and the following result for any value of $n \geq 4$.

> **Theorem 3.4 (Figueroa-Centeno *et al.* [7]):** A cycle C_n, $n \geq 3$, is super-edge-magic if and only if n is odd. ■

Remark 3.5: We see that the class of strongly indexable graphs is a proper subclass of the class of all super-edge magic graphs. Therefore, Theorem 2.5 is a finer result than that of Enomoto *et al.* **[8]**, which establishes that every graph can be embedded in a connected super-edge-magic graph.

Furthermore, Theorem 2.1 gives a characterization of strongly indexable unicyclic graphs and they all happen to be in the class U_3 (*cf.* Acharya and Hegde **[2]**). On the other hand, Figueroa-Centeno *et al.* **[9]** raised the problem of determining which unicyclic graphs are super-edge-magic. This remains an open problem in general. However, in view of Theorem 2.1 and Remark 3.5, we have arrived at a partial solution to the following open problem.

Problem 3.6 [9]: Characterize graphs in the class U_n, $n \geq 3$, that are super-edge-magic.□

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Articles published in *Graph Theory Notes of New York* are of the following types:

(1) Short contributions, surveys, and research articles in graph theory, related fields, and applications.

(2) Graph theory problems (clean or dirty), partial solutions to problems.

(3) Information about availability of graph theory information such as, preprints and technical reports.

(4) Developments on articles in previous *Graph Theory Notes of New York.*

Graph Theory Notes of New York is intended to provide for informal exchange of ideas within the graph theory community. Thus, a good selection of references should be included in each submission.

Please submit contributions to the Editors following the format of a recent issue.

All manuscripts submitted to *Graph Theory Notes of New York* are refereed. Acceptance is the responsibility of the Editors. Accepted articles will not necessarily be published in the order they are received but with the objective of balancing the content of each issue with respect to diversity of topics and authors.

Preferred articles should be concise, self-contained with respect to definitions, provide adequate background to the subject matter. In general, proofs should be in outline form rather than detailed. Papers containing open problems or partial solutions are especially welcome.

Excessive use of figures is discouraged. If used, art work must be submitted in camera ready form (good quality ink drawings or 1200 dpi laser printer output are acceptable). Any text used as items contained in the art work should be in Times Roman and any variables used should be italicized. Please note that figures may be subjected to considerable size reduction and this should be allowed for when preparing the originals.

Electronic submissions are welcome but must be accompanied by a printed copy of the manuscript.

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