GRAPH THEORY NOTES OF NEW YORK

LIX

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Editors:
John W. Kennedy
Louis V. Quintas

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Graph Theory Notes of New York publishes short contributions and research articles in graph theory, its related fields, and its applications.

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This issue includes papers presented at

Graph Theory Day 59

held at

Department of Mathematics and Center for Excellence in Mathematics and Science

Southern Connecticut State University

New Haven, Connecticut

Saturday, May 8, 2010

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INTRODUCTORY REMARKS

We are pleased to announce an enhancement to the editorial structure of *Graph Theory Notes of New York* with the appointment of the following Associate Editors:

Krystyna T. Balińska (Technical University of Poznań, POLAND)
Ivan Gutman (University of Kragujevac, YUGOSLAVIA)
Linda Lesniak (Drew University and Western Michigan University, U.S.A)
Peter J. Slater (University of Alabama at Huntsville, U.S.A)

We appreciate their contributions while they were members of our Editorial Board and look forward to working with them in their new role.

We move ahead in many ways, but somethings remain the same. We continue to ask for your support of the Mathematical Association of America (MAA) Graph Theory Fund. This is part of the sponsorship for *Graph Theory Notes* and Graph Theory Days provided by the Metropolitan New York Section (METRO-NY) of the MAA. Contributions are welcome and can be sent to either of the Editors at their institutional address with the contribution payable to the ”MAA Graph Theory Fund”.

Another ongoing need is that of hosts for Graph Theory Days. Although not trivial, this is a very manageable task that provides a great service to the graph theory community and promotion for the host institutions. In May 2010 a successful Graph Theory Day 59 was hosted by Southern Connecticut State University, New Haven, Connecticut.

All efforts to keep our field of Graph Theory the exciting activity that it is are important. Thus, we call on graph theory enthusiasts to consider submitting an article to *Graph Theory Notes of New York*, contributing to the MAA Graph Theory Fund, and hosting a Graph Theory Day at their institution.

Thank you.

JWK/LVQ
New York
November 2010

GRAPH THEORY DAY 59

Organizing Committee

Joseph E. Fields and Val Pinciu (Southern Connecticut State University)
John W. Kennedy (Queens College, CUNY), Louis V. Quintas (Pace University)

Graph Theory Day 59 was sponsored by The Metropolitan New York Section of The Mathematical Association of America. The event was hosted by the Department of Mathematics and Center for Excellence in Mathematics and Science, Southern Connecticut State University, New Haven, Connecticut. Conference hosts Joseph E. Fields and Val Pinciu introduced the featured speakers. John W. Kennedy and Louis V. Quintas chaired the contributed talks session.

The featured presentations at Graph Theory Day 59 were:

Path Covering of Faulty Hypercubes

Ivan Gotchev
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New Britain, Connecticut, U.S.A.

On Graph Pebbling Numbers and Graham's Conjecture

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The Mathematical Association of America.

[GTN LIX:1] THE UPPER STEINER NUMBER OF A GRAPH

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Abstract

For a connected graph $G = (V, E)$ of order at least 2 and a nonempty set W of vertices in G, the Steiner distance $d(W)$ is the minimum size of a connected subgraph of G containing W. Each such subgraph is a tree and is called a Steiner W-tree. A set $W \subseteq V$ is called a Steiner set of G if every vertex of G is contained in a Steiner W-tree of G. The Steiner number $s(G)$ of G is the minimum cardinality of its Steiner sets and any Steiner set of cardinality $s(G)$ is a minimum Steiner set of G. A Steiner set W in a connected graph G is called a minimal Steiner set if no proper subset of W is a Steiner set of G. The upper Steiner number $s^+(G)$ of G is the maximum cardinality of a minimal Steiner set of G. The upper Steiner numbers of certain classes of graphs are determined. Graphs G of order p with $s^+(G) = p$ or $p - 1$ are characterized. It is shown that for positive integers r, d, and $l \geq 2$, with $r < d \leq 2r$, there exists a connected graph G of radius r, diameter d, and upper Steiner number l. It is also shown that for every two integers a and b such that $2 \leq a \leq b$, there exists a connected graph G with $s(G) = a$ and $s^+(G) = b$.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected, connected graph with no loop or multiple edge. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to **[1][2]**. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x—y path in G. It is known that the distance is a metric on the vertex set of G. An x—y path of length $d(x, y)$ is called an *x—y geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $\text{rad}(G)$, and the maximum eccentricity is the *diameter*, $\text{diam}(G)$, of G. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a Steiner W-tree. It is noted that $d(W) = d(u, v)$ when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a *Steiner set* for G. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set of G and this cardinality is the *Steiner number*, $s(G)$, of G. The Steiner number of a graph was introduced and studied in **[3]**. When $w = \{u, v\}$, every Steiner W-tree in G is a u—v geodesic. Also, $S(W)$ is equal to the set of all vertices lying in u—v geodesics, inclusive of u and v. Hence, Steiner trees, Steiner sets, and Steiner numbers can be considered as extensions of geodesic concepts.

For the graph G shown in Figure 1,

$$W_1 = \{v_1, v_4, v_5\},$$
$$W_2 = \{v_2, v_4, v_7\}, \text{ and}$$
$$W_3 = \{v_3, v_5, v_7\}$$

are the only three s-sets of G, so that $s(G) = 3$.

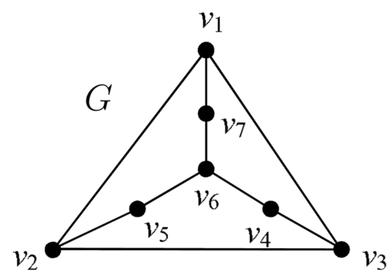

Figure 1: Graph *G*

Two Steiner W_1 trees in the graph G of Figure 1 are shown in Figure 2.

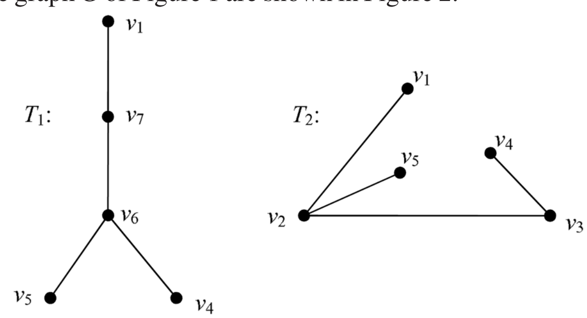

Figure 2: Two Steiner W_1 trees in graph *G*.

For a cut vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v, together with all edges joining v to $V(H)$ is called a *branch* of G at v. An *end block* of G is a block containing exactly one cut vertex of G. Thus, every end block is a branch of G. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by the neighbors of v is a complete graph. Throughout the following G denotes a connected graph with at least two vertices.

The following theorems are used in the sequel.

Theorem 1.1 [3]: Each extreme vertex of a graph G belongs to every Steiner set of G. In particular, each end vertex of G belongs to every Steiner set of G. ■

Theorem 1.2 [3]: Every non-trivial tree with exactly k end vertices has Steiner number k. ■

Theorem 1.3 [3]: For a connected graph G, $s(G) = p$ if and only if $G \cong K_p$. ■

Theorem 1.4 [3]: Let G be a connected graph of order $p \geq 3$.
Then $s(G) = p - 1$ if and only if G contains a cut vertex of degree $p - 1$. ■

2. The Upper Steiner Number of a Graph

Definition 2.1: A Steiner set W in a connected graph G is called a *minimal Steiner set* if no proper subset of W is a Steiner set of G. The *upper Steiner number* $s^+(G)$ of G is the maximum cardinality of a minimal Steiner set of G. ■

Example 2.2: For the graph G shown in Figure 3 (*left*), $W_1 = \{v_1, v_3, v_4\}$ and $W_2 = \{v_1, v_3, v_5\}$ are the only two s-sets, so that $s(G) = 3$. Also, $W = \{v_2, v_4, v_5, v_6\}$ is a minimal Steiner set of G. It is easily verified that no 5-element subset of V is a minimal Steiner set, hence, $s^+(G) = 4$.

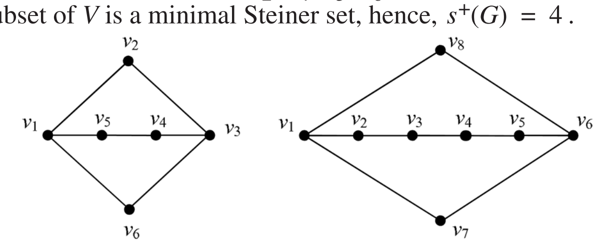

Figure 3:

For the graph G shown in Figure 3 (*right*), $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_1, v_2, v_5\}$, $S_3 = \{v_2, v_5, v_6\}$, and $S_4 = \{v_2, v_3, v_6\}$ are minimum Steiner sets of G so that $s(G) = 3$.

Also $W_1 = \{v_3, v_4, v_7, v_8\}$, $W_2 = \{v_1, v_3, v_5, v_6\}$, $W_3 = \{v_3, v_2, v_4, v_6\}$, $W_4 = \{v_2, v_3, v_5, v_7, v_8\}$, and $W_5 = \{v_2, v_4, v_5, v_7, v_8\}$ are minimal Steiner sets of G, so that $s^+(G) \geq 5$. It is easily verified that no 6-element subset and no 7-element subset of V is a minimal Steiner set of G, and thus $s^+(G) = 5$.

Remark 2.3: Every minimum Steiner set of G is a minimal Steiner set of G, but the converse is not true. For the graph G shown in Figure 3 (*left*), $W = \{v_2, v_4, v_5, v_6\}$ is a minimal Steiner set but not a minimum Steiner set of G.

Theorem 2.4: For a connected graph G of order p, $2 \leq s(G) \leq s^+(G) \leq p$.

Proof: Since any Steiner set needs at least two vertices, $s(G) \geq 2$. Let W be a minimum Steiner set of G, so that $s(G) = |W|$. Since W is also a minimal Steiner set of G, it is clear that $s^+(G) \geq |W| = s(G)$. Since G is a connected graph of order at least two, it contains a spanning tree and thus V is always a Steiner set for G. Hence, $s^+(G) \leq p$. Thus, $2 \leq s(G) \leq s^+(G) \leq p$. ■

Remark 2.5: By Theorem 1.2, for any non-trivial tree T, the set of all end vertices of T is the unique minimum Steiner set of T and thus $s(T) = s^+(T)$. It follows from Theorem 1.3 that $s(G) = 2$ for the complete graph K_2 and that $s^+(G) = p$ for the complete graph K_p ($p \geq 2$). Thus, the bounds in Theorem 2.4 are sharp.

Also, for the graph shown in Figure 2 (*left*), $s(G) = 3$ and $s^+(G) = 4$ so that strict inequality can hold in Theorem 2.4.

Theorem 2.6: For a connected graph G of order p, $s(G) = p$ if and only if $s^+(G) = p$.

Proof: Let $s^+(G) = p$. Then V is the unique minimal Steiner set of G. Since no proper subset of V is a Steiner set, it is clear that V is the unique minimum Steiner set of G and hence $s(G) = p$. The converse follows from Theorem 2.4. ■

Corollary 2.7: For a connected graph G of order p, the following statements are equivalent:
(1) $s(G) = p$,
(2) $s^+(G) = p$, and
(3) $G \cong K_p$ ($p \geq 2$).

Proof: The statement follows from Theorem 1.3 and Theorem 2.6. ■

Theorem 2.8: Let G be a connected graph with v a cut vertex of G and let W be a Steiner set of G. Then every component of $G - v$ contains an element of W.

Proof: Let v be a cut vertex of G and W a Steiner set of G. Suppose that there exists a component, say G_1 of $G - v$ such that G_1 contains no vertex of W. By Theorem 1.1, W contains all the extreme vertices of G and it follows that G_1 does not contain any extreme vertex of G. Thus, G_1 contains at least one edge, say xy. Since every Steiner W-tree T must have its end vertices in W and v is a cut vertex of G, it is clear that vertices x and y do not lie on any Steiner W-tree of G. This contradicts that W is a Steiner set of G. ■

Corollary 2.9: If v is a cut vertex of a connected graph G and W is a Steiner set of G, then v lies in every Steiner W-tree of G. ■

Corollary 2.10: Let G be a connected graph with cut vertices and let W be a Steiner set of G. Then, every branch of G contains an element of W. ■

Corollary 2.11: If G is a connected graph with $k \geq 2$ end blocks, then $s^+(G) \geq k$. ■

Theorem 2.12: No cut vertex of a connected graph G belongs to any minimal Steiner set of G.

Proof: Suppose that there exists a minimal Steiner set W that contains a cut vertex v of G. Let $G_1, G_2, \ldots, G_r$ ($r \geq 2$) be the components of $G - v$. By Theorem 2.8, each component G_i ($i = 1, \ldots, r$) contains an element of W. We claim that $W' = W - \{v\}$ is also a Steiner set of G. Since v is a cut vertex of G, by Corollary 2.9, each Steiner W-tree contains v. Now, since $v \notin W'$, it follows that each Steiner W-tree is also a Steiner W'-tree of G. Thus, W is a Steiner set of G such that $W' \subseteq W$, which is a contradiction to W is a minimal Steiner set of G. Hence, the theorem. ■

Corollary 2.13: For any tree T with k end vertices, $s(T) = s^+(T) = k$.

Proof: This follows from Theorems 1.1 and 2.12. ■

Theorem 2.14: For a complete bipartite graph $K_{m,n}$,

(1) $s^+(K_{m,n}) = 2$ if $m = n = 1$.

(2) $s^+(K_{m,n}) = n$ if $n \geq 2$ and $m = 1$.

(3) $s^+(K_{m,n}) = \max\{m, n\}$ if $m, n \geq 2$.

Proof: Statements (1) and (2) follow from Corollary 2.13.

To prove statement (3), first assume that $m < n$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of $G \cong K_{m,n}$. Let $W = Y$. We prove that W is a minimal Steiner set of G.

Any Steiner W-tree T is a star centered at each x_i ($1 \leq i \leq m$ with y_j ($1 \leq j \leq n$) as the end vertices of T. Hence, every vertex of G lies on a Steiner W- tree, so that W is a Steiner set of G. Let $W' \subset W$. Then there exists a vertex $y_j \in W$ such that $y_j \notin W'$. Since every Steiner W'-tree is a star centered at x_i ($1 \leq i \leq m$) whose end vertices are elements of W', the vertex y_j does not lie on any Steiner W'-tree and thus W' is not a Steiner set of G. This shows that W is a minimal Steiner set of G. Hence, $s^+(G) \geq n$. It can be proved similarly that $W = X$ is also a minimal (in fact, minimum) Steiner set of G.

Let S be any minimal Steiner set of G such that $|S| \geq n + 1$. Hence, S is neither contained in X nor in Y. Furthermore, since X and Y are (minimal) Steiner sets of G, it follows that S contains neither X nor Y. Hence, there exist vertices $x_i \in X$ ($1 \leq i \leq m$) and $y_j \in Y$ ($1 \leq j \leq n$) such that $x_i \notin S$ and $y_j \notin S$. Because the subgraph induced by S is connected, it follows that any Steiner S-tree contains only the vertices of S. Thus, vertices x_i and y_j do not lie on any Steiner S-tree of G, so that S is not a Steiner set of G, which is a contradiction. Thus, any minimal Steiner set of G contains at most n elements, and hence, $s^+(G) \leq n$. Consequently, $s^+(G) = n$. For $m = n$, it can be proved similarly that $s^+(G) = m = n$. ■

Theorem 2.15: If G is a connected, non-complete graph of order p, with no cut vertex, then $s^+(G) \leq p - 2$.

Proof: Suppose that $s^+(G) \geq p - 1$. Then $s^+(G) = p - 1$ or p. If $s^+(G) = p$, then from Corollary 2.7 G is complete, which is a contradiction. Therefore, $s^+(G) = p - 1$. Let v be a vertex of G such that $S = V - \{v\}$ is a minimal Steiner set of G. Since v is not a cut vertex of G, then $\langle S \rangle$ is connected. Hence, S is not a Steiner set of G, which is a contradiction. Thus, $s^+(G) \leq p - 2$. ■

Remark 2.16: The bound in Theorem 2.15 is sharp. For a complete bipartite graph $K_{2,n}$ ($n \geq 2$), it follows from Theorem 2.14 that $s^+(K_{2,n}) = n$.

Theorem 2.17: For a connected graph G, $s(G) = p - 1$ if and only if $s^+(G) = p - 1$.

Proof: Let $s(G) = p - 1$. Then it follows from Theorem 2.4 that $s^+(G) = p$ or $p - 1$. If $s^+(G) = p$, then by Theorem 2.6, $s(G) = p$, which is a contradiction. Hence, $s^+(G) = p - 1$. Conversely, let $s^+(G) = p - 1$. Then it follows from Corollary 2.7 that G is a non-complete graph. Hence, by Theorem 2.15, G contains a cut vertex, say v. Since $s^+(G) = p - 1$, it follows from Theorem 2.12 that $S = S - \{v\}$ is the unique minimal Steiner set of G. Since every minimum Steiner set is also a minimal Steiner set of G, we see that $s(G) = p - 1$. ■

Theorem 2.18: Let G be a connected graph of order $p \geq 3$. Then the following statements are equivalent:

(1) $s(G) = p - 1$.

(2) $s^+(G) = p - 1$.

(3) G contains a cut vertex of degree $p - 1$.

Proof: This follows from Theorems 1.4 and 2.17. ■

3. Realization Results

For every connected graph, $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$. Ostrand **[4]** showed that every two positive integers a and b, with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the upper Steiner number can also be prescribed, when $a \leq b \leq 2a$.

Theorem 3.1: For positive integers *r*, *d*, and $l \geq 2$, with $r \leq d \leq 2r$, their exists a connected graph *G* with $\text{rad}(G) = r$, $\text{diam}(G) = d$, and $s^+(G) = l$.

Proof: When $r = 1$, let $G \cong K_{1,l}$. Then $d = 2$ and, by Corollary 2.13, $s^+(G) = l$.

Now, let $r \geq 2$. We construct a graph *G* with the desired property as follows. Let $C_{2r}: v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order 2*r* and let $P_{d-r+1}: u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let *H* be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} with u_0 in P_{d-r+1}. Now, add $(l-2)$ new vertices $w_1, w_2, \ldots, w_{l-2}$ to *H* and join each vertex w_i ($1 \leq i \leq l-2$) to the vertex u_{d-r-1}, and also join the vertices v_r and v_{r+2} to obtain the graph *G* as illustrated in Figure 4.

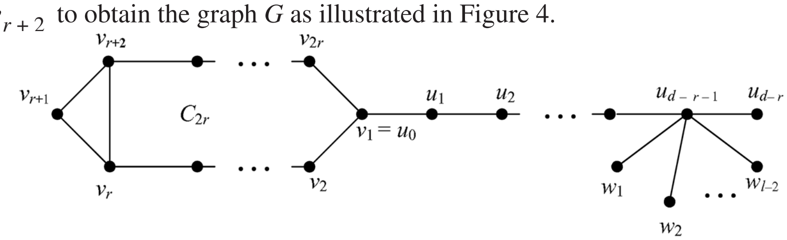

Figure 4:

Then $\text{rad}(G) = r$ and $\text{diam}(G) = d$. Let $W = \{v_{r+1}, w_1, w_2, \ldots, w_{l-2}, u_{d-r}\}$ be the set of all extreme vertices of *G*. By Theorem 1.1, *W* is a subset of every Steiner set of *G*. It is clear that *W* is a Steiner set of *G* and it follows that *W* is the unique minimal Steiner set of *G* so that $s^+(G) = l$. ■

In view of Theorem 2.4, the following theorem gives a realization for the Steiner number and the upper Steiner number of a graph.

Theorem 3.2: For positive integers *a* and *b*, with $2 \leq a \leq b$, there exists a connected graph *G* such that $s(G) = a$ and $s^+(G) = b$.

Proof: If $a = b$, let $G \cong K_{1,a}$. Then, by Corollary 2.13, $s(G) = s^+(G) = a$. If $2 = a < b$, let $G \cong K_{2,b}$. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of *G*. Any Steiner *X*-tree *T* of *G* is a star centered at each y_j ($1 \leq j \leq n$) with x_1 and x_2 as end vertices of *T* and so *X* is a Steiner set of *G*. Since $|X| = 2$, it follows that *X* is a minimum Steiner set, so that $s(G) = 2 = a$. Also, by Theorem 2.14, $s^+(G) = b$.

If $2 < a < b$, let *G* be the graph, illustrated in Figure 5, obtained from the path *P* on three vertices u_1, u_2, u_3, by adding the new vertices $v_1, v_2, \ldots, v_{b-a+1}$ and $w_1, w_2, \ldots, w_{a-2}$, and joining each v_i ($1 \leq i \leq b-a+1$) with u_1 and u_3, and also joining each w_i ($1 \leq i \leq a-2$) with u_1 and u_2.

Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$ be the set of extreme vertices of *G*. Let *S* be any Steiner set of *G*. Then by Theorem 1.1, $W \subseteq S$. It is clear that *W* is not a Steiner set of *G*. Also, it is easily verified that $W \cup \{v\}$, where $v \notin W$, is not a Steiner set of *G*. It is clear that $W \cup \{u_1, u_3\}$ is a Steiner set of *G* and so $s(G) = a$.

Now, let $T = W \cup \{v_1, v_2, \ldots, v_{b-a+1}, u_3\}$. Then it is clear that *T* is a Steiner set of *G*. We show that *T* is a minimal Steiner set of *G*. Let *T′* be any proper subset of *T*. Then there exists at least one vertex, say $u \in T$ such that $u \notin T'$.

If $u = w_i$ for some *i* ($1 \leq i \leq a-2$), then by Theorem 1.1, *T* is not a Steiner set of *G*. If $u = v_j$ for some *j* ($1 \leq j \leq b-a+1$), then the vertex v_j does not lie on any Steiner *T′*-tree of *G*. Similarly, if $u = u_3$, then the vertices u_2 and u_3 do not lie on any Steiner *T′*-tree of *G*. Thus *T′* is not a Steiner set of *G*. Hence, *T* is a minimal Steiner set of *G*, so that $s^+(G) \geq |T| = b$.

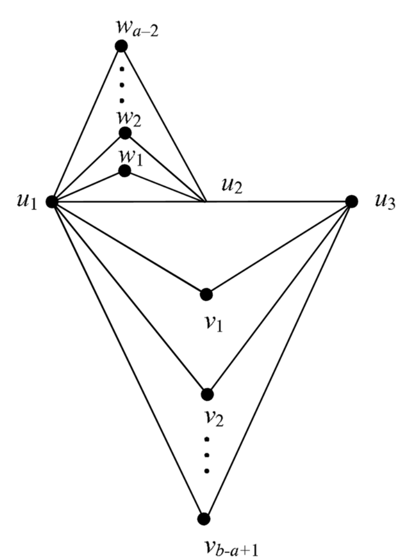

Figure 5:

Now, we show that there is no minimal Steiner set *X* of *G* with $|X| \geq b+1$. Since *G* has $b+2$ vertices and *T* is a Steiner set of *G* with cardinality *b*, it follows that *V* is not a minimal Steiner set of *G*. Suppose that there exists a minimal Steiner set *X* such that $|X| = b+1$. Now, if $u_3 \notin X$, then it is clear that u_3 is not contained in any Steiner *X*-tree and so *X* is not a Steiner set of *G*, which is a contradiction. If $u_3 \in X$, then, since *T* is a

Steiner set of cardinality b and X is minimal, there exists exactly one vertex v_i ($1 \le i \le b-a+1$) such that $v_i \notin X$. It is now clear that v_i is not contained in any Steiner X-tree, and therefore, X is not a Steiner set of G, which is a contradiction. Thus, there is no minimal Steiner set X of G with $|X| \ge b+1$. Hence, $s^+(G) = b$. ■

Remark 3.3: The graph G shown in Figure 5 contains precisely two minimal Steiner sets, viz:

$$W \cup \{u_1, u_3\} \text{ and } T = W \cup \{v_1, v_2, \ldots v_{b-a+1}, u_3\}.$$

Hence, this example shows that there is no intermediate value theorem for minimal Steiner sets. That is, if k is an integer such that $s(G) < k < s^+(G)$, then there need not exist a minimal Steiner set of cardinality k in G.

Using the structure of the graph G constructed in the proof of Theorem 3.2, we can obtain a graph G_n of order n with $s(G_n) = 3$ and $s^+(G_n) = n-1$ for all $n \ge 5$. This suggests the following theorem.

Theorem 3.4: There is an infinite sequence $\{G_n\}$ of connected graphs G_n of order $n \ge 5$ such that $s(G_n) = 3$, $s^+(G_n) = n-1$, $\lim_{n \to \infty} s(G_n)/n = 0$ and $\lim_{n \to \infty} s^+(G_n)/n = 1$.

Proof:

Let G_n be the graph, illustrated in Figure 6, obtained from the path P: u_1, u_2, u_3 of length 2, by adding the new vertices v_1, v_2, v_{n-3}, and w_1, and joining each v_i ($1 \le i \le n-3$) with u_1 and u_3, and also joining w_1 with u_1 and u_2.

Let $W = \{w_1\}$ and $T = \{w_1, v_1, v_2, \ldots, v_{n-3}, u_3\}$. It is clear from the proof of Theorem 3.2 that the graph G_n contains precisely two minimal Steiner sets, namely

$$\{w_1, u_1, u_3\} \text{ and } \{w_1, v_1, v_2, \ldots, v_{n-3}, u_3\},$$

so that $s(G_n) = 3$ and $s^+(G_n) = n-1$.

Hence, the result follows. ■

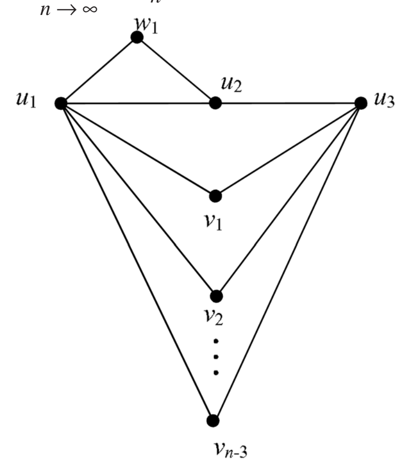

Figure 6:

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ON GRAPH PEBBLING NUMBERS AND GRAHAM'S CONJECTURE

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Abstract

We investigate various results concerning pebbling numbers and optimal pebbling numbers of a connected graph. We imagine discrete pebbles placed on the vertices of a graph. We allow pebbling moves in which two pebbles are removed from some vertex, and one pebble is added to an adjacent vertex. The pebbling number of a graph is the smallest number of pebbles required to ensure that no matter how they are originally placed, we can reach every vertex by a sequence of pebbling moves. The optimal pebbling number of a graph is the smallest number of pebbles for which some placement allows us to reach every vertex. We also discuss Graham's conjecture, which asserts that the pebbling number of a Cartesian product is at most the product of the pebbling numbers of the graphs in that product.

1. Basic Notions

For a graph $G = (V, E)$, a function $D: V \to \mathbb{N}$ is called a *distribution on the vertices* of G, or a *distribution on G*. We usually imagine that $D(v)$ *pebbles* are placed on v for each vertex $v \in V$. For example, Figure 1 shows a distribution of pebbles on P_4, the path with four vertices.

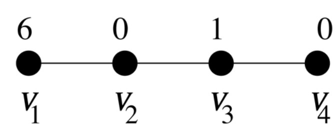

Figure 1: A distribution of pebbles on P_4.

A *pebbling move* in G consists of removing two pebbles from a vertex $v \in V$ that contains at least two pebbles, moving one of these pebbles onto a neighbor of v, and discarding the other. For example, from the distribution shown in Figure 1, we could make two pebbling moves from v_1 to v_2. We would then have two pebbles on v_2 (and two pebbles remaining on v_1). We could then move a second pebble onto v_3, and from there, we could move a pebble onto v_4. Therefore, from this distribution of seven pebbles, it is possible to reach every vertex in P_4. However, if we start with all seven pebbles placed on v_1, it would be impossible to move a pebble onto v_4.

Let $|D|$ denote the *size* of the pebble function D; that is, $|D| = \Sigma_{v \in V} D(V)$. For two distributions D and D' on G, we say that D *contains* D' if $D'(v) \le D(v)$ for all $v \in V$. For two distributions D_1 and D_2, we say that D_2 is *reachable* from D_1 if there is some sequence of pebbling moves beginning with D_1 and resulting in a distribution that contains D_2. We say the distribution D is *solvable*, (respectively, *t-solvable*), if every distribution with one pebble (respectively, t pebbles) on a single vertex is reachable from D.

The traditional *pebbling number*, and *t-pebbling number* of a graph G, denoted by $\pi(G)$ and $\pi_t(G)$, respectively, were defined by Chung **[1]**.The *optimal pebbling number* and *optimal t-pebbling number* of G, denoted by $\pi^*(G)$ and $\pi_t^*(G)$, respectively, were defined by Pacther, Snevily, and Voxman **[2]**. We give those definitions now.

Definitions [1][2]: The *t-pebbling number* of a graph G, denoted by $\pi_t(G)$, is the smallest number such that every distribution with $|D| \ge \pi_t(G)$ is t-solvable. The *optimal t-pebbling number* of G, denoted by $\pi_i^*(G)$, is the smallest number such that some distribution with $\pi_t(G)$ pebbles is t-solvable. In both cases we omit the t when $t = 1$. Thus, the *pebbling number* of G is $\pi(G) = \pi_1(G)$ and the *optimal pebbling number* of G is $\pi^*(G) = \pi_1^*(G)$. ■

We leave the reader to verify that $\pi(P_4) = 8$ and $\pi^*(P_4) = 3$. Note the following basic result.

Proposition 1.1: For any graph G with n vertices and diameter $D(G)$, $\pi^*(G) \le n \le \pi(G)$ and $\pi^*(G) \le 2^{D(G)} \le \pi(G)$.

Proof: Placing one pebble on every vertex creates a solvable distribution, thus $\pi^*(G) \le n$. Placing one pebble on every vertex except for some vertex v creates an unsolvable distribution with $n-1$ pebbles: v is unreachable because no move is possible. Therefore, $\pi(G) \ge n$. Similarly, placing $2^{D(G)}$ pebbles on any vertex v creates a solvable distribution, but if the distance from v to w is $D(G)$, placing $2^{D(G)} - 1$ pebbles on v gives a distribution from which w is unreachable. Therefore, $\pi^*(G) \le 2^{D(G)} \le \pi(G)$. ■

2. Graham's Conjecture

Graham's conjecture asserts a bound on the pebbling number of the Cartesian product of two graphs.

Definition: If $G = (V, E)$ and $G' = (V', E')$ are two graphs, their *Cartesian product* is the graph $G \times G'$ with vertex set:

$$V(G \times G') = V \times V' = \{(x, x') : x \in V, x' \in V'\}$$

and whose edge set is:

$$E(G \times G') = \{((x, x'), (y, x')) : (x, y) \in E\} \cup \{((x, x'), (x, y')) : (x', y') \in E'\}.$$ ■

To provide examples, Figure 2 shows a graph G together with the products $G \times K_2$ and $G \times K_3$, where K_n represents the complete graph on n vertices. We also write G^d for the graph $G \times G \times \ldots \times G$, the product of d copies of G.

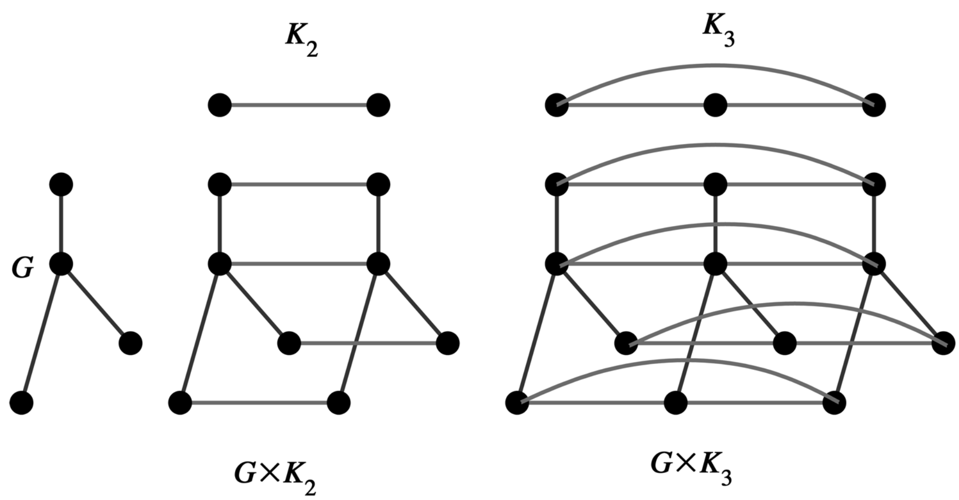

Figure 2: Cartesian products.

Chung **[1]** attributed the following conjecture to Graham.

Conjecture 2.1 (Graham's Conjecture): For graphs G and G', $\pi(G \times G') \le \pi(G)\pi(G')$. □

Conjecture 2.2 is a generalization of Conjecture 2.1.

Conjecture 2.2: For any multiset of graphs $\{G_1, G_2, \ldots, G_k\}$,

$$\pi(G_1 \times G_2 \times \ldots \times G_k) \le \pi(G_1)\pi(G_2)\ldots\pi(G_k).$$ □

Conjectures 2.1 and 2.2 appear very difficult to resolve in general, but they have been proved for some specific graphs.

Theorem 2.3 [1][3][4]: For a multiset of graphs $\{G_1, G_2, \ldots, G_k\}$, if every G_i is a complete graph, a tree, or a cycle, and at most two of the G_i are 5-cycles, then Conjecture 2.2 holds. That is, $\pi(G_1 \times G_2 \times \ldots \times G_k) \le \pi(G_1)\pi(G_2)\ldots\pi(G_k)$. ■

Cartesian products involving C_5 turn out to be difficult to work with. It is known (from **[5]**) that $\pi(C_5) = 5$ and $\pi(C_5 \times C_5) = 25$, but it has turned out to be difficult to verify the following conjecture, due to Chung **[1]**.

Conjecture 2.4 [1]: $\pi(C_5 \times C_5 \times C_5) = 125$ and $\pi(C_5^d) = 5^d$. □

3. Two-Pebbling Properties

It is natural to ask at this point whether it is possible to say anything at all, in general, about Graham's conjecture. For example, it is easy to see that $\pi(K_n) = n$ for any positive integer *n*. Is it possible to guarantee that $\pi(G \times K_n) \le n\pi(G)$ for any graph **G**? The answer, in general, is no, but we can guarantee this for any graph that satisfies one of two possible two-pebbling properties.

To motivate a description of these properties, we ask how might we prove inductively that $\pi(G \times K_n) \le n\pi(G)$. Assume that $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Further assume the inductive hypothesis that $\pi(G \times K_m) \le m\pi(G)$ when $m < n$ (the base case $n = 1$ is trivial). Suppose we have an arbitrary distribution of $n\pi(G)$ pebbles on $G \times K_n$. We may assume, without loss of generality, that the target vertex is (x_0, v_1) for some $x_0 \in V(G)$. Suppose we try to transfer $(n-1)\pi(G)$ pebbles onto the vertices of the form (x, v_i), with $i \le n-1$. Denote these vertices by $V(G) \times \{v_1, v_2, \ldots, v_{n-1}\}$. We transfer as many pebbles as possible from each (x, v_n) to (x, v_1). Unfortunately, if (x, v_n) has an odd number of pebbles, we must leave a pebble behind.

These ideas motivate the following notation: given a distribution $D: V(G \times K_n) \to \mathbb{N}$ with $|D| = n\pi(G)$, let p_i be the number of pebbles on $V(G) \times \{v_i\}$ and let r_i be the number of vertices in $V(G) \times \{v_i\}$ with an odd number of pebbles. Then, by transferring pebbles from each (x, v_n) to (x, v_1), we can ensure a total of $p_1 + p_2 + \ldots + p_{n-1} + (p_n - r_n)/2$ pebbles on $V(G) \times \{v_1, v_2, \ldots, v_{n-1}\}$. If it is not possible to reach the target (x_0, v_1) from there, it must be the case that $p_1 + p_2 + \ldots + p_{n-1} + (p_n - r_n)/2 < (n-1)\pi(G)$. Since we started with $p_1 + p_2 + \ldots + p_n = n\pi(G)$ pebbles on the graph, this would mean that $(p_n + r_n)/2 > \pi(G)$ or equivalently, $p_n + r_n > 2\pi(G)$. From these distributions, we try to move two pebbles onto (x_0, v_n). If we succeed, we could then move a pebble onto (x_0, v_1), as desired. The two-pebbling properties are designed to allow this approach to succeed when the first strategy fails. Chung **[1]** first defined the two-pebbling property and Wang **[6]** defined the odd two-pebbling property.

Definitions [1][6]: Given a distribution of pebbles on the vertices of a graph *G*, let $p = |D|$, let *q* be the number of occupied vertices in *G*, and let *r* be the number of vertices with an odd number of pebbles. We say *G* satisfies the *two-pebbling property* if it is possible to move two pebbles to any vertex of *G* by a sequence of pebbling moves whenever $p + q > 2\pi(G)$. We say *G* satisfies the *odd two-pebbling property* if it is possible to move two pebbles to any vertex of *G* by a sequence of pebbling moves whenever $p + r > 2\pi(G)$. ■

The previous argument shows that if *G* satisfies the odd two-pebbling property, then $G \times K_n$ satisfies Graham's conjecture, $\pi(G \times K_n) \le \pi(G)\pi(K_n) = n\pi(G)$. Theorem 3.1 offers a few more examples of products that satisfy Graham's conjecture.

Theorem 3.1: If *G* satisfies the odd two-pebbling property, then $\pi(G \times H) \le \pi(G)\pi(H)$ provided *H* is a complete graph, a complete bipartite graph, a tree, or an even cycle graph. ■

Note that if *G* satisfies the two-pebbling property, it automatically satisfies the odd two-pebbling property: if $p + r > 2\pi(G)$, then $p + q \ge p + r > 2\pi(G)$. Therefore, if *G* satisfies the two-pebbling property, two pebbles can be moved to any vertex. The following conjecture asserts the converse.

Conjecture 3.2: *G* satisfies the two-pebbling property if and only if it satisfies the odd two-pebbling property. □

Many graphs satisfy at least the odd two-pebbling property.

Theorem 3.3 [1]–[3]: *G* satisfies the odd two-pebbling property if the diameter of *G* is at most two (e.g., complete graphs and complete bipartite graphs), or if *G* is a tree or a cycle. ■

A natural question to ask is whether any graph does not satisfy these properties. We call such a graph a *Lemke graph*, in honor of Paul Lemke who discovered the example L shown in Figure 3. It takes some effort, but one can verify that $\pi(L) = 8$. However, for the given distribution, we have $p = 13$ and $q = r = 5$, but two pebbles cannot be moved to v. Lemke graphs are thought to be the most likely candidates for counterexamples to Graham's conjecture. It would be interesting to verify the following conjecture.

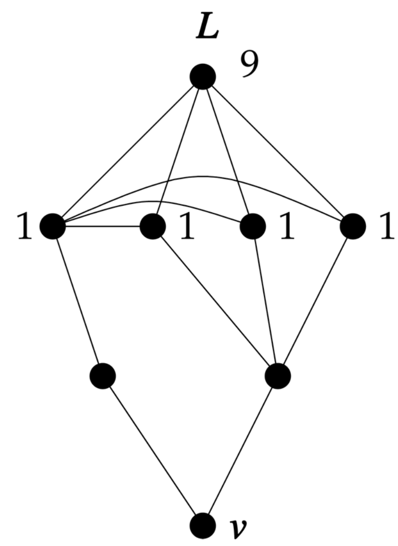

Figure 3: The Lemke graph *L*.

Conjecture 3.4: If L is the Lemke graph shown in Figure 3, then $\pi(L \times L) = 64$. □

4. A Generalization of Graham's Conjecture

We can generalize what we allow as the target distributions, without insisting that all pebbles have to be on the same vertex. This gives us more general versions of pebbling numbers.

Definitions: If D is any distribution of pebbles on the vertices of the graph G, we say the distribution D' is D-solvable if it is possible to go from D' to a distribution that contains D by a sequence of pebbling moves. We then define $\pi(G, D)$ as the smallest number such that every distribution of $\pi(G, D)$ pebbles on G is D'-solvable. Similarly, if S is a set of distributions of pebbles on G, we define $\pi(G, S)$ as the smallest number such that every distribution of $\pi(G, S)$ pebbles on G is D-solvable for every $D \in S$. We also define $\pi^*(G, S)$ to be the smallest number such that some distribution of $\pi(G, S)$ pebbles on G is D-solvable for every $D \in S$. ■

In particular, if we define $S_t(G)$ as the set of distributions with t pebbles on a single vertex, then $\pi(G, S_t(G)) = \pi_t(G)$ and $\pi^*(G, S_t(G)) = \pi_t^*(G)$. For a single distribution D, $\pi_t^*(G)$ is not interesting; since D is D-solvable, and trivially $\pi^*(G, D) = |D|$.

To generalize Graham's conjecture, we generalize products to work with sets of distributions.

Definitions: Given distributions $D_G\colon V(G) \to \mathbb{N}$ and $D_H\colon V(H) \to \mathbb{N}$ on the vertices of graphs G and H, respectively, we define $D_G \cdot D_H\colon V(G \times H) \to \mathbb{N}$ as the distribution on $G \times H$ given by $(D_G \cdot D_H)((v, w)) = D_G(v)D_H(w)$.
If S_G and S_H are sets of distributions on G and H, respectively, we define $S_G \cdot S_H$ as the set of distributions on $G \times H$ given by $S_G \cdot S_H = \{D_G \cdot D_H : D_G \in S_G, D_H \in S_H\}$. ■

Figure 4 illustrates an example of the product of a distribution on the graph G from Figure 2 and a distribution on K_2.

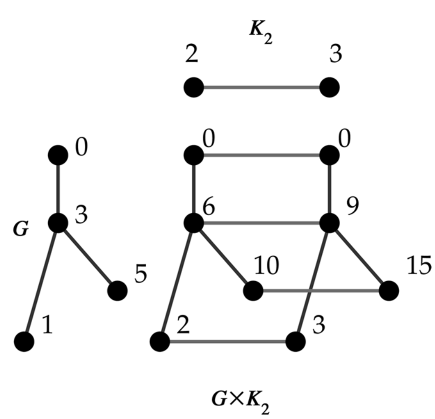

Figure 4: An example of product distribution.

We can now generalize Conjecture 2.1 as follows:

Conjecture 4.1: For any graphs G and H, and any sets of distributions S_G and S_H on the vertices of G and H, respectively,

$$\pi(G \times H, S_G \cdot S_H) \le \pi(G, S_G)\pi(H, S_H)\,. \qquad \square$$

Note that if $S_G = S_1(G)$ and $S_H = S_1(H)$, then $S_G \cdot S_H = S_1(G \times H)$, hence with this choice of distributions, we obtain Graham's conjecture precisely.

Obviously, Conjecture 4.1 is very difficult for regular pebbling. However, the analog for optimal pebbling follows relatively easily from the following nontrivial observations.

Observations: If we are able to get from the distribution D_1' to D_1 in G by a sequence of pebbling moves and we can get from D_2' to D_2 in H, then we can get from $D_1' \cdot D_2'$ to $D_1' \cdot D_2$ to $D_1 \cdot D_2$ in $G \times H$ by a sequence of pebbling moves. In particular, if we can get from D_G to every distribution in S_G, and we can get from D_H to every distribution in S_H, then we can get from $D_G \cdot D_H$ to every distribution in $S_G \cdot S_H$.

Since $|D_G \cdot D_H| = |D_G||D_H|$, Theorem 4.2 follows from the above observations. This theorem generalizes a result from Shiue **[7]** that $\pi^*_{st}(G \times H) \le \pi^*_s(G)\pi^*_t(H)$.

Theorem 4.2 [8]: For any graphs G and H, and any sets of distributions S_G and S_H on the vertices of G and H, respectively, then $\pi^*(G \times H, S_G \cdot S_H) \le \pi^*(G, S_G)\pi^*(H, S_H)$. ■

Thus, for optimal pebbling, an analog of Graham's conjecture holds in a very general setting.

5. Some Solvable Distributions

In this section, we construct solvable distributions for hypercubes and for Cartesian products of C_5. These are not optimal distributions, at least not for large products, but they are better then previously known distributions.

5.1. Solvable Distributions on Hypercubes

Write Q^d for the d-dimensional *hypercube*, $Q^d \cong K_2^d$, and label the vertices of Q^d by bitstrings. Given a vertex $v \in Q^d$ and a bit $b \in \{0, 1\}$, let $v \cdot b$ be the vertex in Q^{d+1} obtained by appending b to the bitstring for v. Moews **[9]** proved the following theorem, which gives the best known bound on the optimal pebbling number of hypercubes.

Theorem 5.1 [9]: $\pi^*(Q^d) \in \left(\frac{4}{3}\right)^{d + O(\log d)} = O\left(\frac{4}{3}^d d^k\right)$ for some constant k. ■

Moews proof, however, was probabilistic; he did not construct explicit distributions. It also does not tell us anything when d is small. The best previously constructed distributions are given in the following theorem, obtained by Pachter, Snevily, and Voxman **[2]**.

Theorem 5.2 [2]: If $d = 2k$, then putting $2k$ pebbles each on the vertices $00\ldots0$ and $11\ldots1$ gives a solvable distribution on Q^d. If $d = 2k+1$, then putting $2k+1$ pebbles on $00\ldots0$ and putting $2k$ pebbles on $11\ldots1$ gives a solvable distribution on Q^d. ■

We call the distributions in Theorem 5.2 *antipodal distributions*, and write A_d for the antipodal distribution on Q^d. The number of pebbles required by A_d is in $\Theta(\sqrt{2}^n) \approx \Theta(1.4142^n)$. We inductively extend the antipodal distributions to produce solvable distributions with fewer pebbles by using the following definition:

Definition: Let $D\colon V(Q^d) \to \mathbb{N}$ be a distribution of pebbles on Q^d, and for each $v_i \in V(Q^d)$, let $p_i = D(v_i)$. Then construct a distribution $\rho(D)\colon V(Q^{d+1}) \to \mathbb{N}$ as follows:

- If $p_i = 3k$, let $(\rho(D))(v_i \cdot 0) = (\rho(D))(v_i \cdot 1) = 2k$.
- If $p_i = 3k+1$, let $(\rho(D))(v_i \cdot 0) = (\rho(D))(v_i \cdot 1) = 2k+1$.
- If $p_i = 3k+2$, let $(\rho(D))(v_i \cdot 0) = 2k+2$ and $(\rho(D))(v_i \cdot 1) = 2k+1$.

We write $p^m(D)$ for the distribution on Q^{d+m} obtained by applying this construction m times. ■

Note that in each case, we can put p_i pebbles on $v_i \cdot b$ for each $b \in \{0, 1\}$. In particular, if D is solvable in Q^d, then $\rho(D)$ is solvable in Q^{d+1}. Note further that $|\rho(D)| \approx \frac{4}{3}|D|$ if each p_i is sufficiently large. Unfortunately, this requirement that p_i must be sufficiently large limits our ability to repeatedly improve the bound by a factor of 4/3, but if we start with a sufficiently large antipodal distribution, a careful analysis, conducted in **[8]** gives the following result.

Theorem 5.3 [8]: Given an integer d, let

$$k = \left\lceil \frac{\log_2 1.5}{\log_2 4.5}(d-1) \right\rceil \approx 0.2696(d-1) \text{, and let } m = d-1-2k \approx 0.4608(d-1) .$$

Then the distribution $\rho^m(A_{2k+1})$ on $Q^{2k+m+1} = Q^d$ satisfies

$\left|\rho^m(A_{2k+1})\right| \in O(2^m) \approx O(1.3763^d)$. ■

5.2. Solvable Distributions on C_5^d

Moews **[9]** generalized Theorem 5.1 to apply to all graphs. In particular, applying it to C_5 gives the following theorem:

Theorem 5.4 [9]: $\pi^*(C_5^d) \in 2^{d+O(\log d)} = O(2^d d^k)$ for some constant k. ■

This theorem gives the best known bound on the optimal pebbling number involving products of C_5, but it suffers from the same drawbacks as Theorem 5.1. It does not construct explicit distributions and it does not tell us anything for small values of d. The first explicit distributions for C_5^d were given in **[5]**. They were based on the partition of $C_5 \times C_5$ implied by the boldfaced edges shown in Figure 5.

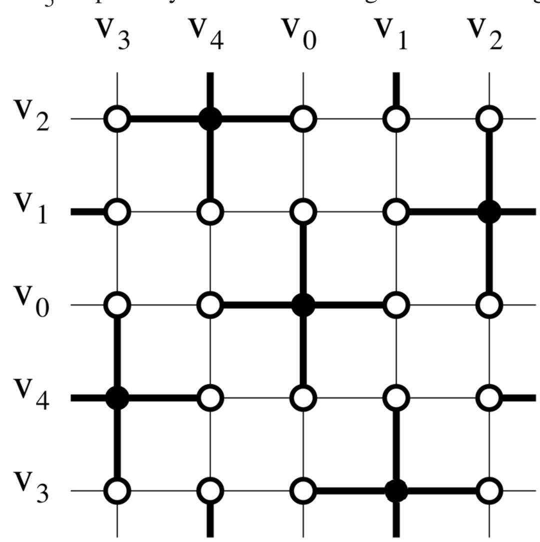

Figure 5: A partition of $C_5 \times C_5$.

In Figure 5, the edges wrap around from the left side to the right side, and from the top to the bottom. The main observation is that if we place four pebbles each on the vertices represented by dark circles, the resulting distribution is 4-solvable. For example, if the target is (v_0, v_1), we can move two pebbles there from (v_0, v_0) and one pebble each from (v_3, v_1) and (v_1, v_2), and by symmetry, every unoccupied vertex can be reached similarly. We call this Distribution B, and we use B in Theorem 5.5, from **[5]**.

Theorem 5.5 [5]: Let G be any graph and let D be a t-solvable distribution on G in which the number of pebbles on every vertex is a multiple of four. Then, the distribution $B \cdot D$ is a t-solvable distribution in $(C_5 \times C_5) \times G$ in which the number of pebbles on every vertex is a multiple of four. Note that the number of pebbles in $B \cdot D$ is $5|D|$. In particular, by induction on k, we have $\pi_i^*(C_5^{2k} \times G) \le 5^k|D|$. ■

Starting with $G \approx C_5$, let D be the 1-solvable distribution with four pebbles on a single vertex. Then $\pi^*(C_5^{2k+1}) \le 4 \cdot 5^k$. For even products C_5^{2k}, a solvable distribution on C_5^4 was given in **[8]** in which eleven vertices each have four pebbles, and no other vertices are occupied (the bound $\pi^*(C_5 \times C_5) \le 8$ is achieved separately). In particular, the following theorem provides an explicit bound on $\pi^*(C_5^d)$.

Theorem 5.6 [8]: $\pi^*(C_5^{2k+1}) \le 4 \cdot 5^k$ and $\pi^*(C_5^{2k}) \le \frac{44}{25}(5^k)$. Thus, for all $d \ge 1$
$\pi^*(C_5^d) \le \frac{4}{\sqrt{5}}(5^{d/2}) \in O(\sqrt{5}^d)$. ■

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[GTN LIX:3] CANONICAL CONSISTENCY OF SIGNED LINE STRUCTURES

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Abstract

A marked signed graph is an ordered pair $S_\mu = (S, \mu)$, where $S = \{S^u, \sigma\}$ is a sigraph and $\mu\colon V(S^u) \to \{+, -\}$ is a function from the vertex set $V(S^u)$ of S^u into the set $\{+, -\}$, called a marking of S. A cycle Z in S_μ is said to be consistent if it contains an even number of negatively marked vertices. A sigraph S is consistent if every cycle in S is consistent. In particular, σ induces a unique marking μ_σ defined by $\mu_\sigma(v) = \Pi_{e_j \in E_v}\sigma(e_j)$, where E_v is the set of edges incident on v in S, and is called the canonical marking of S. In this paper, we characterize canonically consistent line sigraphs and canonically consistent $\times$-line sigraphs, the latter being a variation of the standard notion of a line sigraph.

1. Introduction

A *signed graph* (or *sigraph*) is an ordered pair $S = (S^u, \sigma)$, where $S^u = (V, E)$ is a graph, called the *underlying graph* of S and $\sigma\colon E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$ called the *signature* of S. Let $E^+(S) = \{e \in E(S) : \sigma(e) = +\}$ and $E^-(S) = \{e \in E(S) : \sigma(e) = -\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative edges* of S, respectively. For standard terminology and notation in graph theory see Harary **[1]** and West **[2]**. For sigraphs see Zaslavsky **[3][4]**. Throughout this paper graphs are finite with no loop or multiple edge.

A sigraph S is called *regular* if the number of positive edges, $d^+(v)$, incident on a vertex v in S and the number of negative edges, $d^-(v)$ incident on v, are the same for all vertices in S, with $d^+(v)$ and $d^-(v)$ not necessarily equal. The numbers $d^+(v)$ and $d^-(v)$ are called the *positive* and *negative degrees* of v in S, respectively. For a regular sigraph S, the degree of S is the pair $(d^+(v), d^-(v))$.

The *edge degree* $d_e(e_j)$ of an edge e_j in a sigraph S is the total number of edges adjacent to e_j in S. The *positive* (*negative*) *edge degree*, $d_e^+(e_j)$ ($d_e^-(e_j)$) of edge e_j in S is the total number of positive (negative) edges adjacent to e_j. The *negation* $\eta(S)$ of a sigraph S is a sigraph obtained from S by negating the sign of every edge of S. Thus, to obtain $\eta(S)$, change the sign of every edge of S to its opposite.

An alternating sequence of vertices and edges of S, beginning and terminating with vertices, in which all the vertices are distinct, is called a *path* in S. The *length* of a path is defined to be the number of edges contained in the path. A path containing precisely n edges, is called an n-path. If all the edges in a path are negative, then the path is called an *all-negative path*. A *negative section* (see **[5]**) of a subsigraph S' of a sigraph S is a maximal edge-induced connected subsigraph in S consisting of only the negative edges of S. The *length* of a negative section is the number of negative edges it contains.

For a sigraph S, Behzad and Chartrand **[6]** defined its *line sigraph* $L(S)$ as the sigraph in which the edges of S are represented by vertices with two vertices adjacent in $L(S)$ whenever the corresponding edges in S share a common vertex. An edge ef in $L(S)$ is defined to be negative whenever both e and f are negative edges in S. In **[7][8]**, the authors introduced a variation of the standard notion for the line sigraph $L(S)$ of a given sigraph S: $L_\times(S)$ is a sigraph defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge ef of $L(S^u)$ the product of signs of the adjacent edges e and f of S. $L_\times(S)$ is called the $\times$-*line sigraph* of S.

A sigraph $S_1 = (S_1^u, \sigma_1)$ is $(S_2, \mathcal{R})$ *-marked* if there exists a sigraph $S_2 = (S_2^u, \sigma_2)$, a bijection $\varphi\colon E(S_2) \to V(S_1^u)$, a binary relation $\mathcal{R}$ on $E(S_2)$, and a marking $\mu\colon V(S_1^u) \to \{+1, -1\}$ of S_1^u satisfying the following *compatibility conditions*:

(1) $uv \in E(S_1^u) \Leftrightarrow \{\varphi^{-1}(u), \varphi^{-1}(v)\} \in \mathcal{R}$,

(2) $\{\mu(u), \mu(v)\} = \{\sigma(\varphi^{-1}(u)), \sigma(\varphi^{-1}(v))\}$, $\forall uv \in E(S_1^u)$.

Furthermore, S_1 is $(S_2, \mathcal{R})$-consistent if the following condition is satisfied:

(3) $\prod_{v \in V(Z)} \mu(v) = 1$, for every cycle Z in S_1.

The case when $\mathcal{R}$ is defined by the condition $\varphi^{-1}(u) \cap \varphi^{-1}(v) = \varnothing$ is treated in Sinha **[9]** in a study of signed graph equations involving signed line graphs. In this particular case, the terms $(S_2, \mathcal{R})$-marked and $(S_2, \mathcal{R})$-consistent will be simplified to S_2-marked and S_2-consistent, respectively. In particular, σ induces a unique marking μ_σ, called the *canonical marking* of S, defined by

$$\mu_\sigma(v) = \prod_{e_j \in E_v} \sigma(e_j),$$

where E_v is the set of edges incident on v in S.

If every vertex of a sigraph S is canonically marked, then a cycle Z in S is said to be *canonically consistent* if Z has an even number of negatively marked vertices. A sigraph S is said to be canonically consistent if every cycle in S is canonically consistent.

2. Canonically Consistent Line Sigraphs

Beineke and Harary **[10][11]** were the first to pose the problem of characterizing consistent marked graphs. This question was subsequently settled by Acharya **[12][13]** and by Hoede **[14]**. Acharya and Sinha obtained S-consistency of line sigraphs in **[9][15]**. In this section, we obtain a characterization of canonically consistent line sigraphs.

The following theorem by Hoede plays an important role in solving this problem.

Theorem 1 [14]: A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles with respect to T are consistent and all common paths of pairs of these fundamental cycles have end vertices carrying the same marks. ■

Theorem 2: The line sigraph $L(S)$ of a sigraph $S = (S^u, \sigma)$ is canonically consistent if and only if the following conditions hold in S:

(1) The number of negative edges of odd negative edge degree is even
(a) for every cycle Z in S, and
(b) for the edges incident at v with $d(v) = 3$ such that v does not lie on any cycle Z in S;

(2) If $d(v) = 3$ and e_i, e_j, e_k are edges incident at v with e_i, e_j lying on the cycle Z, then the numbers of all-negative 2-paths from e_i and e_j have the same parity and the number of all-negative 2-paths from e_k is even;

(3) If $d(v) > 3$, then for all negative edges e_j incident at v, $d_e^-(e_j) \equiv 0 \pmod 2$.

Proof: <u>Necessity</u> Suppose $L(S)$ is canonically consistent, then every cycle Z' in $L(S)$ is canonically consistent. Thus, Z' must have an even number of negatively marked vertices. Let $Z = v_1 e_1 \ldots v_n e_n v_1$ be a cycle in S. By definition of $L(S)$, $Z' = e_1 e_2 \ldots e_n e_1$ is a cycle in $L(S)$. Clearly, a positive edge and a negative edge of even negative edge degree in Z will result in positively marked vertices in Z' but a negative edge of odd negative edge degree in Z will create a negatively marked vertex in Z'. Since Z' has an even number of negatively marked vertices, this means that the number of negative edges of odd negative edge degree is even. Thus, (1a) follows.

Next, let $d(v) = 3$. Suppose v does not lie on any cycle Z in S and that the edges e_i, e_j, e_k are incident at v. Since these three edges form a triangle in $L(S)$ and $L(S)$ is canonically consistent, the number of negative edges of odd negative edge degree amongst them must be even. Thus, (1b) follows.

Now, let $d(v) = 3$ and suppose that e_i, e_j, e_k are edges incident at v with e_i and e_j lying on the cycle Z. By definition of $L(S)$, the edge e_ie_j in $L(S)$ is in the cycles Z' and Z'' created, respectively, from the vertex of degree three and the cycle Z. Because $L(S)$ is canonically consistent, using Theorem 1, we obtain

(1) $\mu_\sigma(e_i) = \mu_\sigma(e_j)$.

Assume that the numbers of all-negative 2-paths from e_i and e_j have opposite parity. Then e_i and e_j are oppositely marked vertices in $L(S)$, a contradiction to equation **(1)**. However, if $\mu_\sigma(e_i) = \mu_\sigma(e_j)$ and Z' is canonically consistent, then e_k must be a positively marked vertex in Z'. Consequently, the number of all-negative 2-paths from e_k is even and (2) follows.

Now, let v be a vertex in S such that $d(v) > 3$ and let e_j be a negative edge incident at v. Suppose that $d_e^-(e_j) \equiv 1 \pmod 2$, then $d_e^-(e_j)$ is odd. Then an odd number of negative edges are adjacent to e_j in S and because of the canonical marking, e_j is a negatively marked vertex in $L(S)$. Thus, this will form an inconsistent cycle in $L(S)$, a contradiction to the assumption that $L(S)$ is canonically consistent. Therefore, $d_e^-(e_j) \equiv 0 \pmod 2$ for all negative edges e_j incident at v, and (3) follows.

Sufficiency Suppose Conditions (1), (2), and (3) hold for a sigraph S. We show that $L(S)$ is canonically consistent,; that is, every cycle Z' in $L(S)$ must have an even number of negatively marked vertices. Let Z' be a cycle in $L(S)$ corresponding to the cycle Z in S. By Condition (1a), Z contains an even number of negative edges of odd negative edge degree, so by the definition of $L(S)$, Z' contains an even number of negatively marked vertices. Now let $d(v) = 3$, with v not lying on any cycle Z in S, and suppose edges e_i, e_j, e_k are incident at v. Clearly, by the definition of $L(S)$, there is a cycle Z' due to the three edges incident at v. By Condition (1b), there is an even number of negative edges of odd negative edge degree incident at v. This means that the cycle Z' contains either two or zero negatively marked vertices.

Now, let $d(v) = 3$ and let e_i, e_j, e_k be edges incident at v, where e_i and e_j lying on the cycle Z in S. By the definition of $L(S)$, the edge e_ie_j in $L(S)$ is common to the cycles Z' and Z'' created by the three edges incident at v and the cycle Z. Now, by Condition (2), the number of all-negative 2-paths from e_i and e_j have the same parity, this means that

$$\mu_\sigma(e_i) = \mu_\sigma(e_j),$$

and again, by Condition (2),

$$\mu_\sigma(e_k) = +.$$

Consequently, Z' contains an even number of negatively marked vertices, by Condition (1a), Z'' contains an even number of negatively marked vertices, and using Theorem 1, the cycle formed by the symmetric difference of Z' and Z'' also contains an even number of negatively marked vertices.

By condition (3), all the vertices of cycles in $L(S)$ due to vertices of degree greater than three in S are positively marked. Hence the theorem. ■

Theorem 3: If for every negative edge e_j in a sigraph S, $d_e^-(e_j) \equiv 0 \pmod 2$, then the line sigraph $L(S)$ of S is canonically consistent.

Proof: Since only negative edges in S generate negatively marked vertices in $L(S)$, $d_e^-(e_j) \equiv 0 \pmod 2$ for every negative edge e_j in S. This implies that an even number of negative edges is adjacent to every negative edge in S, and hence, by definition of $L(S)$, every vertex in $L(S)$ is marked positively. Thus $L(S)$ is canonically consistent. ■

Corollary 4: If a sigraph S has all-negative sections of length at most one, then the line sigraph $L(S)$ of S is canonically consistent. ■

Corollary 5: Theorem 3 and Corollary 4 provide sufficient conditions for a line sigraph $L(S)$ of S to be canonically consistent. ■

Corollary 6: The line sigraph $L(S)$ of a regular sigraph S is canonically consistent. ■

Corollary 7: If for every positive edge e_j in a sigraph S, $d_e^-(e_j) \equiv 0 \pmod 2$ then the line sigraph $L(\mu(S))$ of $\mu(S)$ is canonically consistent. ■

3. Canonically Consistent ×-Line Sigraphs

In this section, we obtain a characterization of canonically consistent ×-line sigraphs.

Theorem 8: The ×-line sigraph $L_\times(S)$ of a sigraph $S = (S^u, \sigma)$ is canonically consistent if and only if the following conditions hold in S:

(1) For every cycle Z in S, the number of negative edges of odd edge degree is even;

(2) If $d(v) > 3$, then for all e_j incident at v,
(a) $d_e^-(e_j) \equiv 0 \pmod 2$ if $\sigma(e_j) = +$, and
(b) $d_e^+(e_j) \equiv 0 \pmod 2$ if $\sigma(e_j) = -$;

(3) If $d(v) = 3$ and e_i, e_j, e_k are edges incident at v with e_i and e_j lying on the cycle Z having $d_e(e_i) = l$, $d_e(e_j) = m$, and $d_e(e_k) = n$; and e_{i_p} $(p = 1, 2, \ldots, l)$, e_{j_q} $(q = 1, 2, \ldots, m)$, and e_{k_r} $(r = 1, 2, \ldots, n)$ are edges adjacent to e_i, e_j, and e_k, respectively, then

(a) $\prod_{p=1}^{l} (\sigma(e_i))^l \sigma(e_{i_p}) = \prod_{q=1}^{m} (\sigma(e_j))^m \sigma(e_{j_q})$, and

(b) $\prod_{r=1}^{n} (\sigma(e_k))^n \sigma(e_{k_r}) = +$;

(4) If $d(v) = 3$ and v does not lie on any cycle Z in S, then the edges e_i, e_j, and e_k incident at v satisfy Conditions (3a) and (3b).

Proof: Necessity Suppose $L_\times(S)$ is canonically consistent, then every cycle Z' in $L_\times(S)$ is canonically consistent. This means that Z' must have an even number of negatively marked vertices. Let $Z = v_1 e_1 \ldots v_n e_n v_1$ be a cycle in S. By definition of $L_\times(S)$, $Z' = e_1 e_2 \ldots e_n e_1$ is a cycle in $L_\times(S)$. Let $d_e(e_j) = l_j$ for each edge e_j, $j = 1, 2, \ldots, n$. Since

$$\textbf{(2)} \qquad \prod_{j=1}^{n} \mu(e_j) = \prod_{j=1}^{n} (\sigma(e_j))^{l_j+2} \prod_{i=1}^{h} (\sigma(e_j'))^2,$$

where $h = (\sum_{j=1}^{n} l_j)/2 - n$ and $e_1', e_2', \ldots, e_h'$ are edges not contained in Z.

If l_j is even, then $l_j + 2$ is also even. Thus the right hand side of Equation **(2)** is always positive irrespective of the signs of edges of the cycle Z. If e_j is positive, then again the right hand side of Equation **(2)** is positive. Thus, let $e_1, e_2, \ldots, e_k$ be the negative edges of odd edge degree and $e_{k+1}, e_{k+2}, \ldots, e_n$ be the positive or negative edges of even edge degree. Then,

$$\textbf{(3)} \qquad \prod_{j=1}^{n} \mu(e_j) = \prod_{j=1}^{k} (\sigma(e_j))^{l_j+2} \left[\prod_{j=k+1}^{n} (\sigma(e_j))^{l_j+2} \prod_{i=1}^{h} (\sigma(e_j'))^2 \right].$$

Since $L_\times(S)$ is canonically consistent, the right hand side of Equation **(3)** must be positive but the factor

$$\left[\prod_{j=k+1}^{n} (\sigma(e_j))^{l_j+2} \prod_{i=1}^{h} (\sigma(e_j'))^2 \right]$$

is positive, so the factor $\prod_{j=1}^{k} (\sigma(e_j))^{l_j+2}$ is also positive.

Since $l_j + 2$ is odd and e_j is a negative edge for $j = 1, 2, \ldots, k$, then k must be even and (1) follows.

Now, let v be a vertex in S such that $d(v) > 3$ and let e_j be a positive edge incident at v. If $d_e^-(e_j) \equiv 1 \pmod 2$, then $d_e^-(e_j)$ is odd so that an odd number of negative edges are adjacent to e_j in S. Then, because of the canonical marking, e_j is a negatively marked vertex in $L_\times(S)$. This creates an inconsistent cycle in $L_\times(S)$, a contradiction to the assumption that $L_\times(S)$ is canonically consistent. Therefore,

$$d_e^+(e_j) \equiv 0 \pmod 2 \text{ if } \sigma(e_j) = -.$$

Hence, (2a) follows.

Now suppose $d(v) = 3$ and let edges e_j, e_j, and e_k be incident at v, with e_i and e_j lying on a cycle Z. Let $d_e(e_i) = l$, $d_e(e_j) = m$, and $d_e(e_k) = n$. By the definition of $L_\times(S)$, the edge e_ie_j in $L_\times(S)$ is common to the cycles Z' and Z'' respectively created the vertex of degree three and the cycle Z. Since $L_\times(S)$ is canonically consistent, and using Theorem 1, we obtain

(4) $$\mu_\sigma(e_i) = \mu_\sigma(e_j).$$

Then

(5) $$\mu_\sigma(e_i) = \prod_{p=1}^{l} \sigma(e_ie_{i_p}) = \prod_{p=1}^{l} (\sigma(e_i))^l\sigma(e_{i_p}).$$

Similarly,

(6) $$\mu_\sigma(e_j) = \prod_{q=1}^{m} (\sigma(e_j))^m\sigma(e_{j_q}).$$

From **(4)**, **(5)**, and **(6)**, we obtain

$$\prod_{p=1}^{l} (\sigma(e_i))^l\sigma(e_{i_p}) = \prod_{q=1}^{m} (\sigma(e_j))^m\sigma(e_{j_q}).$$

Thus, (3a) follows.

Next, since Z' is a cycle in $L_\times(S)$ resulting from the edges e_i, e_j, and e_k, and $\mu_\sigma(e_i) = \mu_\sigma(e_j)$, then

$$\mu_\sigma(e_k) = \prod_{r=1}^{n} (\sigma(e_k))^n\sigma(e_{k_r}) = +,$$

otherwise we have an inconsistent cycle Z' in $L_\times(S)$. Thus, (3b) follows.

Now let $d(v) = 3$, where v does not lie on any cycle Z in S, and let edges e_i, e_j, e_k be incident at v. Since these three edges form a triangle in $L_\times(S)$, Condition (3a) and (3b) must be satisfied by edges e_i, e_j, and e_k, otherwise, there is a contradiction to the assumption. Hence, (4) follows.

Sufficiency Suppose conditions (1), (2), (3), and (4) hold for a sigraph S. We show that $L_\times(S)$ is canonically consistent; that is, every cycle Z' in $L_\times(S)$ has an even number of negatively marked vertices. Let Z' be a cycle in $L_\times(S)$ corresponding to the cycle Z in S. By condition (1), Z contains even number of negative edges of odd edge degree. Using Equation **(3)**, Z' contains an even number of negatively marked vertices. By condition (2), any cycle in $L_\times(S)$ resulting from a vertex of degree greater than three in S contains an even number of negatively marked vertices.

Now suppose $d(v) = 3$ and let e_i, e_j, and e_k be edges incident at v with e_i and e_j lying on the cycle Z, with $d_e(e_i) = l$, e$d_e(e_j) = m$, and $d_e(e_k) = n$. By the definition of $L_\times(S)$, the edge e_ie_j in $L_\times(S)$ is common to the cycles Z' and Z'' respectively created by the three edges incident at v and the cycle Z. By Condition (3a), $\mu_\sigma(e_i) = \mu_\sigma(e_j)$ and by Condition (3b), $\mu_\sigma(e_k) = +$. Consequently, Z' contains an even number of negatively marked vertices. By Condition (1), Z'' contains an even number of negatively marked vertices. Thus, using Theorem 1, the cycle formed by the symmetric difference of Z' and Z'' also contains an even number of negatively marked vertices.

Now let $d(v) = 3$ where v does not lie on a cycle and suppose the edges e_i, e_j, and e_k are incident at v. Clearly, by definition of $L_\times(S)$ there is a cycle Z' due to the three edges incident at v. By Condition (4), the edges e_i, e_j, and e_k satisfy Conditions (3a) and (3b). Thus, the cycle Z' contains two or zero negatively marked vertices. Hence the theorem. ■

Theorem 9: If for every edge e_j in a sigraph S,

(1) $d_e^-(e_j) \equiv 0 \pmod 2$ if $\sigma(e_j) = +$ and

(2) $d_e^+(e_j) \equiv 0 \pmod 2$ if $\sigma(e_j) = -$,

then the $\times$-line sigraph $L_\times(S)$ of S is canonically consistent.

Proof: If e_j is positive edge in S, then by condition (1), an even number of negative edges is adjacent to e_j in S. This implies that e_j is a positively marked vertex in $L_\times(S)$. If e_j is a negative edge in S, then by condition (2) there is an even number of positive edges adjacent to e_j in S. This means that e_j is a positively marked vertex in $L_\times(S)$. Thus $L_\times(S)$ is canonically consistent. ■

Remark 10: Theorem 9 provides a sufficient condition for a $\times$-line sigraph $L_\times(S)$ of a sigraph S to be canonically consistent.

Corollary 11: The $\times$-line sigraph $L_\times(S)$ of a regular sigraph S is canonically consistent. ■

Theorem 12: If the $\times$-line sigraph $L_\times(S)$ is canonically consistent for a sigraph S, then $L_\times(\eta(S))$ is also canonically consistent. ■

4. Conclusion

In this paper, we obtained the characterization of canonically consistent line sigraphs and canonically consistent $\times$-line sigraphs. We have also derived various sufficient conditions for a line sigraph and a $\times$-line sigraph to be canonically consistent.

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[GTN LIX:4] ENUMERATION OF HAMILTON CYCLES AND TRIANGLES IN EULER TOTIENT CAYLEY GRAPHS

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Abstract
Hamilton cycles are cycles of largest length and triangles are cycles of smallest length in a graph. In this paper the number of Hamilton cycles and triangles in a class of Cayley graphs associated with the Euler totient function $\phi(n)$, for integer $n \geq 1$, are determined.

1. Introduction

Berrizbeitia and Giudici **[1][2]** and Dejter and Giudici **[3]** studied the cycle structure of *Cayley graphs* associated with certain arithmetic functions. In this paper we determine the number of Hamilton cycles and triangles in a class of Cayley graphs associated with the *Euler totient function* $\phi(n)$. The enumeration of Hamilton cycles and triangles in quadratic residue Cayley graphs is presented elsewhere **[4]**.

Let $(X, \cdot)$ be a group. A subset *S* of *X* is called a *symmetric subset* provided that for all $s \in S$, $s^{-1} \in S$. The graph *G* with vertex set *X* and edge set $\{gh : g^{-1}h \in S \text{ or } hg^{-1} \in S\}$ is called the *Cayley graph* of *X* corresponding to the symmetric set *S*. We denote this graph by $G(X, S)$. Clearly, $G(X, S)$ is an undirected graph that contains no loop if the identity element *e* of *X* is deleted from *S*. It is easy to see that the Cayley graph $G(X, S)$ is $|S|$-regular and that the size of $G(X, S)$ is $|X||S|/2$.

2. Euler Totient Cayley Graph and Its Properties

For positive integer *n*, let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the set of equivalence classes modulo *n*. Then $(\mathbb{Z}_n, \oplus)$ is an Abelian group of order *n*, where $\oplus$ denotes addition modulo *n*. Let *S* denote the set of all positive integers that are less than *n* and relatively prime to *n*. Then, $|S| = \phi(n)$, the *Euler totient function*. It is easy to see that *S* is a symmetric subset of the group $(\mathbb{Z}_n, \oplus)$; and that $(S, \odot)$ is a multiplicative subgroup with order $\phi(n)$ of the semigroup $(\mathbb{Z}_n^*, \odot)$, where $\mathbb{Z}_n^* = \mathbb{Z}_n - \{0\}$, and $\odot$ denotes multiplication modulo *n*.

Definition 2.1: For positive integer *n*, let $(\mathbb{Z}_n, \oplus)$ be the additive group of integers modulo *n* and let *S* be the set of all positive integers less than *n* and relatively prime to *n*. The *Euler totient Cayley graph* $G(\mathbb{Z}_n, \Phi)$ is defined as the graph whose vertex set is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and whose edge set is $\{xy : x - y \in S \text{ or } y - x \in S\}$. ■

Because the graph $G(\mathbb{Z}_n, \Phi)$ is the Cayley graph of the group $(\mathbb{Z}_n, \oplus)$ associated with the symmetric set *S*, the following lemma is immediate.

Lemma 2.2: The graph $G(\mathbb{Z}_n, \Phi)$ is $\phi(n)$-regular. Moreover, the size of $G(\mathbb{Z}_n, \Phi)$ is $n\phi(n)/2$. ■

Lemma 2.3: The graph $G(\mathbb{Z}_n, \Phi)$ is Hamiltonian, and hence, it is connected.

Proof: Let *s* be an element of *S*. Then $0 < s < n$ and *s* is relatively prime to *n*. Hence, s is a generator of for $(\mathbb{Z}_n, \oplus)$. Consequently, $s, 2s, \ldots, (n-1)s$ are all distinct and $\{s, 2s, \ldots, ns\} = \mathbb{Z}_n$. For $1 < r < n$, $(r+1)s - rs = s \in S$. Thus, for each *r*, $1 \leq r \leq n$, there is an edge connecting $(r+1)s$ and rs. Consequently, $G(\mathbb{Z}_n, \Phi)$ contains the Hamilton cycle $(0, s, 2s, \ldots, ns = 0)$. Thus $G(\mathbb{Z}_n, \Phi)$ is Hamiltonian, and thus, is a connected graph. ■

Definition 2.4: For $s \in S$, the cycle $C_s = (0, s, 2s, \ldots, ns = 0)$ is called the *Hamilton cycle* corresponding to the element *s* in *S*. ■

Lemma 2.5: For $n \geq 3$, the graph $G(\mathbb{Z}_n, \Phi)$ is Eulerian.

Proof: The graph $G(\mathbb{Z}_n, \Phi)$ is $\phi(n)$-regular. By Theorem 2.5(e) of **[5]**, $\phi(n)$ is even for $n \geq 3$. Thus, the degree of each vertex in $G(\mathbb{Z}_n, \Phi)$ is even so that $G(\mathbb{Z}_n, \Phi)$ is Eulerian. ■

Theorem 2.6: If *n* is even, then the graph $G(\mathbb{Z}_n, \Phi)$ is bipartite.

Proof: We show that $G(\mathbb{Z}_n, \Phi)$ has no odd cycle. To see this, let $(i_1, i_2, \ldots, i_r, i_1)$ be a cycle in $G(\mathbb{Z}_n, \Phi)$. Then $i_1 i_2, i_2 i_3, \ldots, i_r i_1$ are edges in $G(\mathbb{Z}_n, \Phi)$, so that $i_s - i_{s+1} \in S$ for $1 \leq s \leq r-1$. Since *n* is even, $i_s - i_{s+1}$ and $i_r - i_1$ are both odd for $1 \leq s \leq r-1$. That is, one of i_s and i_{s+1} is even and the other is odd for $1 \leq s \leq r-1$, and the same is true for i_1 and i_r. Thus, $i_1, i_2, \ldots, i_r, i_1$ alternate in parity. This shows that half of $i_1, i_2, \ldots, i_r$ are even and the other half are odd. Consequently, their number (*r*) is even. It follows that the cycle $(i_1, i_2, \ldots, i_r, i_1)$ is an even cycle. Hence, $G(\mathbb{Z}_n, \Phi)$ has no odd cycle, so that **[6]** the graph $G(\mathbb{Z}_n, \Phi)$ is bipartite. ■

Corollary 2.7: If *n* is even, then $G(\mathbb{Z}_n, \Phi)$ has no triangle.

Proof: By Theorem 2.6, if *n* is even, then $G(\mathbb{Z}_n, \Phi)$ has no odd cycle. Hence, $G(\mathbb{Z}_n, \Phi)$ has no triangle. ■

3. Enumeration of Disjoint Hamilton Cycles

Lemma 3.1: For any $s \in S$, the Hamilton cycles associated with *s* and with $n - s$ are the same.

Proof: Let *s* be an element of *S*. Then, by Lemma 2.3, the graph $G(\mathbb{Z}_n, \Phi)$ has a Hamilton cycle:

$$C_s = (0, s, 2s, \ldots, (n-2)s, (n-1)s, ns = 0).$$

In the Abelian group $(\mathbb{Z}_n, \oplus)$, $nt = 0$ for $1 \leq t \leq n$. Hence, for any *r*, $0 \leq r \leq n$,

$$(n-r)s = ns - rs = 0 - rs = rn - rs = r(n-s).$$

Thus the cycle $C_{n-s} = (0, (n-s), 2(n-s), \ldots, (n-1)(n-s), 0)$ is the same as C_s. ■

Lemma 3.2: For $s, t \in S$, $t \neq s$, and $t \neq n - s$, the Hamilton cycles C_s and C_t are edge disjoint.

Proof: Let $s, t \in S$ such that $t \neq s$ and $t \neq n - s$. Then the Hamilton cycles C_s and C_t are

$$C_s = (0, s, 2s, \ldots, (n-1)s, ns = 0), \text{ and } C_t = (0, t, 2t, \ldots, (n-1)t, nt = 0).$$

By Lemma 3.1,

$$\begin{aligned} C_s &= (0, s, 2s, \ldots, (n-1)s, ns = 0) \\ &= (0, (n-s), 2(n-s), \ldots, (n-1)(n-s), n(n-s) = 0) = C_{n-s}. \end{aligned}$$

We claim that the Hamilton cycles C_s and C_t are edge disjoint. Suppose that C_s and C_t are not edge disjoint. Then there exists an edge $(it, (i+1)t)$ in C_t such that either

$$(it, (i+1)t) = (js, (j+1)s) \text{ or } (it, (i+1)t) = (k(n-s), (k+1)(n-s))$$

for some $0 \leq j \leq n-1$ or $0 \leq k \leq n-1$.

However, $(it, (i+1)t) = (js, (j+1)s)$ implies that $it = js$ and $(i+1)t = (j+1)s$, which is a contradiction. Similarly, $(it, (i+1)t) = (k(n-s), (k+1)(n-s))$ implies that $it = k(n-s)$ and $(i+1)t = (k+1)(n-s)$, which is also a contradiction. Therefore, the Hamilton cycles C_s and C_t are edge disjoint. ■

Theorem 3.3: For $n \geq 3$, the Euler totient graph $G(\mathbb{Z}_n, \Phi)$ can be decomposed into $\phi(n)/2$ edge-disjoint Hamilton cycles.

Proof: Let $n \geq 3$ be an integer. We show that $s \neq n - s$ for all $s \in S$. If $s = 1$, then $n - s = n - 1 \geq 2$. Since $n \geq 3$, then $n - 1 \neq 1$. On the other hand if $s \neq 1$ and $s = n - s$, then $n = 2s$, so that $\gcd(s, n) = \gcd(s, 2s) = s$. Since $s \neq 1$, this contradicts the fact that $\gcd(s, n) = 1$ because $s \in S$. Consequently, $s \neq n - s$ for all $s \in S$.

Hence, S can be partitioned into $\phi(n)/2$ disjoint pairs of distinct numbers, $(s, n - s)$. By Lemma 3.1, the Hamilton cycles corresponding to each pair are the same. Thus, by Lemma 3.2, these $\phi(n)/2$ distinct pairs produce $\phi(n)/2$ edge disjoint Hamilton cycles.

Since each Hamilton cycle contains $|\mathbb{Z}_n| = n$ edges, the total number of edges contributed by the $\phi(n)/2$ edge-disjoint Hamilton cycles is $|\mathbb{Z}_n|\phi(n)/2$ and, by Lemma 2.2, this is equal to the total number of edges in the $G(\mathbb{Z}_n, \Phi)$.

Hence, the graph $G(\mathbb{Z}_n, \Phi)$ can be decomposed into $\phi(n)/2$ edge-disjoint Hamilton cycles. ■

4. Enumeration of Triangles

In this section we obtain a formula for the number of triangles in $G(\mathbb{Z}_n, \Phi)$ in terms of another well known arithmetic function, namely the *Schemmel totient function* $\phi^{(2)}(n)$, which denotes the number of pairs of consecutive positive integers that are less than *n* and both relatively prime to *n* (see **[7]**).

Because $G(\mathbb{Z}_n, \Phi)$ is a Cayley graph, it is vertex transitive. Thus, to count the number of triangles we concentrate our attention on triangles of the form $(0, a, b)$, where $a, b \in S$ and $(a - b) \in S$. Trivially, $1 \in S$. Hence, for any $b \in S$, the triple $(0, 1, b)$ is a triangle if $(b - 1) \in S$. A triangle of this form is called a *fundamental triangle*. We denote the set of all fundamental triangles by Δ_{01}. That is,

$$\Delta_{01} = \{(0, 1, b) : b \in S \text{ and } (b - 1) \in S\}.$$

Theorem 4.1: If $n \geq 3$ is an odd integer, then $|\Delta_{01}| = \phi^{(2)}(n)$.

Proof: Let *n* be an odd positive integer, $n \geq 3$. Then the triple $(0, 1, b)$ is a fundamental triangle if and only if $b \in S$ and $b - 1 \in S$, which implies that *b* and $b - 1$ are consecutive numbers that are less than *n* and relatively prime to *n*. Thus, there are as many fundamental triangles in $G(\mathbb{Z}_n, \Phi)$ as there are pairs of consecutive positive integers that are less than *n* and relatively prime to *n*. Hence, $|\Delta_{01}| = \phi^{(2)}(n)$. ■

Definition 4.2: For each $\mu \in S$, define $\Delta_\mu = \{(0, \mu, k) : k, (k - \mu) \in S\}$.
That is, Δ_μ is the set of all triangles of the form $(0, \mu, k)$, for $\mu \in S$. ■

Theorem 4.3: For each $\mu \in S$, $|\Delta_\mu| = |\Delta_{01}| = \phi^{(2)}(n)$.

Proof: We claim that the mapping $f\colon \Delta_{01} \to \Delta_\mu$ given by $f(0, 1, b) = (0, \mu, \mu b)$ is a bijection.

To see this, let $(0, \mu, \mu b_1) = (0, \mu, \mu b_2)$ for some $b_1, b_2 \in S$. Then $\mu b_1 = \mu b_2$. Since $(S, \odot)$ is a group, then $b_1 = b_2$. Consequently, $(0, 1, b_1) = (0, 1, b_2)$, and *f* one-to-one.

Let $(0, \mu, k)$ be any element of Δ_μ. Then, μ, k, and $(k - \mu)$ are all in *S*. For $k, \mu \in S$, there is a unique element *b* in *S* such that $k = \mu b$. Thus, $(k - \mu) \in S$ implies that $(\mu b - \mu) \in S$ or $\mu(b - 1) \in S$. Hence, $b - 1 \in S$ and $(0, 1, b) \in \Delta_{01}$. However, $(0, \mu, k) = (0, \mu, \mu b) = f(0, 1, b)$. Thus the function *f* is onto and consequently *f* is a bijection. Therefore, $|\Delta_\mu| = |\Delta_{01}| = \phi^{(2)}(n)$. ■

Theorem 4.4: Let $\Delta(0)$ denote the set of all triangles with 0 as one vertex. Then, for any odd integer $n \geq 3$,

$$|\Delta(0)| = \frac{1}{2}\phi(n) \cdot \phi^{(2)}(n).$$

Proof: By definition $\Delta(0) = \{(0, \mu, k) : \mu, k, (k - \mu) \in S\}$. For fixed $\mu \in S$, clearly

$$\Delta(0) = \bigcup_{\mu \in S} \Delta_\mu.$$

The triangle $(0, \mu, k)$ appears twice in the union, once in Δ_μ and once in Δ_k. Hence, by Theorem 4.3

$$
\begin{aligned}
|\Delta(0)| &= \frac{1}{2}\sum_{\mu \in S} |\Delta_\mu| \\
&= \frac{1}{2}\sum_{\mu \in S} \phi^{(2)}(n) \\
&= \frac{1}{2}\phi^{(2)}(n) \cdot |S| = \frac{1}{2}\phi(n) \cdot \phi^{(2)}(n).
\end{aligned}
$$

■

Theorem 4.5: For any integer $n \geq 3$ the total number of triangles $T(\Phi)$ in $G(\mathbb{Z}_n, \Phi)$ is given by

$$
T(\Phi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{6}n\phi(n)\phi^{(2)}(n) & \text{if } n \text{ is odd.} \end{cases}
$$

Proof: If n is even, then by Corollary 2.7, $T(\Phi) = 0$. Thus, let n be an odd positive integer. $G(\mathbb{Z}_n, \Phi)$ is vertex transitive and $\phi(n)$-regular. Thus, the same number of triangles passes through each vertex and this number is $\frac{1}{2}\phi(n)\phi^{(2)}(n)$. Therefore, because $G(\mathbb{Z}_n, \Phi)$ has order n, the total number of triangles in $G(\mathbb{Z}_n, \Phi)$ is $\frac{1}{2}n\phi(n)\phi^{(2)}(n)$.

However, each triangle in $G(\mathbb{Z}_n, \Phi)$ is counted three times (once for each of its three vertices). Thus, the number $T(\Phi)$ of distinct triangles in $G(\mathbb{Z}_n, \Phi)$ is given by

$$
T(\Phi) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{6}n\phi(n)\phi^{(2)}(n) & \text{if } n \text{ is odd.} \end{cases}
$$

■

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[GTN LIX:5] ON FRACTIONAL EFFICIENT DOMINATING SETS OF GRAPHS

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Abstract

A dominating function or a fractional dominating set of a graph $G = (V, E)$ is a function $f: V \to [0, 1]$ such that for all $v \in V$, the sum of the function values over the closed neighborhood of v is at least one. A minimal dominating function or a fractional minimal dominating set f is a fractional dominating set such that f is not a fractional dominating set if for any $v \in V$ the value of $f(v)$ can be decreased. A function f is called a fractional independent set if $\Sigma_{x \in N[v]} f(x) = 1$, for every vertex v with $f(v) > 0$. An independent function f is called a maximal fractional independent set, if $\Sigma_{x \in N[v]} f(x) \geq 1$ for every $v \in V$ with $f(v) = 0$. A set $S \subseteq V$ is a perfect dominating set if for every vertex $v \in V - S$, $|N(v) \cap S| = 1$. A perfect dominating set S is minimal, if no proper subset of S is a perfect dominating set. We define the fractional version of minimal perfect dominating set and make a comparative study of various subsets of the fractional minimal dominating sets of a graph.

1. Introduction

In this paper $G = (V, E)$ represents an undirected graph with no loop or multiple edge. For graph theoretic terms that are not defined here refer to West **[1]**. In this section we discuss basic definitions and results. A subset D of the vertex set $V(G)$ of a graph G, is called a *dominating set* if $N[v] \cap D \neq \emptyset$ for any $v \in V - D$, where $N[v]$ is the *closed neighborhood* of vertex v. A dominating set D is a *minimal dominating set*, if no proper subset of D is a dominating set. More than one thousand research papers have been published on approximately one hundred variations of domination in graphs. Almost all versions of dominating sets and their properties, are discussed by Haynes, Slater, and Hedetniemi in **[2]**. Another text by the same authors **[3]**, discusses almost all advanced topics in this area and contains many comprehensive survey articles. The fractional version of a dominating set is popularly known as a *dominating function* of a graph $G(V, E)$ and is defined as a function $f: V \to [0, 1]$ such that

$$\sum_{x \in N[v]} f(x) \geq 1$$

for all $v \in V$. A *minimal dominating function* is a dominating function f such that f is not a dominating function if for any $v \in V$ the value of $f(v)$ is decreased. In this paper, we call these functions *fractional dominating sets* and *fractional minimal dominating sets*, respectively. For any dominating function $f: V \to [0, 1]$ of G,

$$f(N[v]) = \sum_{x \in N[v]} f(x).$$

The *boundary* of f denoted by B_f is

$$\left\{ v \in V : \sum_{x \in N[v]} f(x) = 1 \right\}.$$

The *positive set* of f, denoted by P_f is $\{v \in V : f(v) > 0\}$. For any two subsets A and B of V, we say that A *dominates* B (denoted by $A \to B$) if each vertex in $B - A$ is adjacent to some vertex in A. The *fractional dominating number* of a graph G, denoted by

$$\gamma_f(G) = \min\left\{\sum_{x \in V} f(x) : f \text{ is a fractional mimimal dominating set of } G\right\}.$$

The set of all fractional minimal dominating sets of a graph G is denoted by $\mathcal{F}_D(G)$. The following theorem provides a way to identify fractional minimal dominating sets of a given graph.

Theorem 1.1 [4]: A fractional dominating set f of a graph G is a fractional minimal dominating set if and only if $B_f \to P_f$. ■

A *convex combination* of two fractional minimal dominating sets f and g of G is $h_\lambda = \lambda f + (1-\lambda)g$, where $0 < \lambda < 1$. Because any convex combination of two fractional dominating sets is a fractional dominating set, the set of all fractional dominating sets is a convex set. However, it is evident from the following theorem that the convex combination of two fractional minimal dominating sets need not always be a fractional minimal dominating set. The next theorem offers a necessary and sufficient condition for a convex combination of two fractional minimal dominating sets to be minimal.

Theorem 1.2 [4]: A convex combination of two fractional minimal dominating sets f and g is minimal if and only if $B_f \cap B_g \to P_f \cup P_g$. ■

In this context Cockayne *et al.* **[4]** introduced the concept of a universal fractional minimal dominating set. A fractional minimal dominating set of G is called a *universal fractional minimal dominating set* if and only if its convex combination with any other fractional minimal dominating set is minimal.

Theorem 1.3 [4]:

1. If the fractional minimal dominating set g satisfies $B_g = V$ and if for any fractional minimal dominating set f, B_f dominates V, then g is a universal fractional minimal dominating set.
2. If g is a universal fractional minimal dominating set, then B_g dominates V. ■

To continue with our characterization of universal fractional minimal dominating sets we require the following definitions.

Definitions: A vertex $v \in V$ is said to *absorb* the vertex $u \neq v$, and u is said to be absorbed by v, if $N[u] \subset N[v]$. In this case, v is called an *absorbing vertex* and u is called an *absorbed vertex*.

Let $A = \{v \in V : v \text{ is an absorbing vertex}\}$ and $\Omega = \{u \in V : u \text{ is an absorbed vertex}\}$. A vertex w of a graph G is said to be *f-sharp*, where f is an fractional minimal dominating set of G, if $B_f \cap N[w] \subseteq A$. Furthermore, w is said to be *sharp* if w is f-sharp, for some fractional minimal dominating set f of G. ■

Lemma 1.4 [4]: Let f be an fractional minimal dominating set of G and let w be an f-sharp vertex. Then

1. If $v \in B_f \cap N[w]$ with $u \in \Omega - A$ a vertex absorbed by v, then $w \notin N[u]$;
2. $w \notin B_f$; and
3. $f(w) = 0$. ■

Theorem 1.5 [5]: The fractional minimal dominating set g of a graph $G = (V, E)$ is a universal fractional minimal dominating set if and only if

1. $V - A \subseteq B_g$ and
2. $g(w) = 0$ for each sharp vertex w of G. ■

Theorem 1.2 can be extended to a finite number of fractional minimal dominating sets. Thus, we obtain the following generalization.

Theorem 1.6 [6]: A convex combination of n fractional minimal dominating sets $f_1, f_2, \ldots, f_n$ is minimal if and only if $B_{f_1} \cap B_{f_2} \cap \ldots \cap B_{f_n} \to P_{f_1} \cup P_{f_2} \cup \ldots \cup P_{f_n}$. ■

The fact that the set of fractional dominating sets is convex and some fractional dominating sets cannot be expressed as a convex combination of two or more fractional dominating sets motivates the definition of basic fractional dominating sets and basic fractional minimal dominating sets **[6]**. A fractional minimal dominating set is called a *basic fractional minimal dominating set* if it cannot be expressed as a proper convex combination of two distinct fractional minimal dominating sets. Kumar *et al.* **[7]** presented a necessary and sufficient condition for a fractional minimal dominating set to be a basic fractional minimal dominating set. Based on this condition they developed an algorithm to determine whether a given fractional minimal dominating set is basic.

Theorem 1.7 [7]: Let *f* be a fractional minimal dominating set. The function *f* is a basic fractional minimal dominating set if and only if there does not exist a fractional minimal dominating set *g* such that $B_f = B_g$ and $P_f = P_g$. ■

Theorem 1.8 [7]: Let *f* be a fractional minimal dominating set of a graph $G = (V, E)$ with $B_f = \{v_1, v_2, \ldots, v_m\}$ and $P_f = \{u \in V : 0 < f(u) < 1\} = \{u_1, u_2, \ldots, u_n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is adjacent to } u_j \text{ or } v_j = u_j \\ 0 & \text{otherwise.} \end{cases}$$

Consider the system of linear equations

(1) $$\sum_j a_{ij} x_j = 0 \text{, for } 1 \le i \le m .$$

Then the function *f* is a basic fractional minimal dominating set if and only if Equation **(1)** has only a trivial solution. ■

Corollary 1.9 [7]: If $f(v) \in [0, 1]$ for all $v \in V$, then the fractional minimal dominating set *f* of *G* is a basic fractional minimal dominating set. ■

A set $S \subseteq V$ is called an *independent set* of a graph *G*, if no pair of vertices of *S* is adjacent in *G*. A *maximal independent set* of *G* is an independent set *S* such that there does not exist another independent set *S′* that properly contains *S*. In **[6]** Kumar suggested a definition for the fractional versions of both independent sets and maximal independent sets.

A function $f\colon V \to [0, 1]$ is called a *fractional independent set* if for every vertex *v* with $f(v) > 0$, $\Sigma_{x \in N[v]} f(x) = 1$. A fractional independent set *f* is called a *maximal fractional independent set* if for every $v \in V$ with $f(v) = 0$, $\Sigma_{x \in N[v]} f(x) \ge 1$. The set of all maximal fractional independent sets is denoted by $\mathcal{F}_I$. We observe that if *S* is (a maximal) an independent set in *G*, then $f = \chi_S$ is (a maximal) a fractional independent set.

Theorem 1.10 [8]: A function *f* is a fractional independent set if and only if $P_f \subseteq B_f$. ■

A convex combination of two maximal fractional independent sets *f* and *g* is defined by $h_\lambda = \lambda f + (1 - \lambda) g$, where $0 < \lambda < 1$. A characterization of convex maximal fractional independent sets is provided by the next result.

Lemma 1.11 [8]: A convex combination of two maximal fractional independent sets *f* and *g* is a maximal fractional independent set if and only if $P_f \cup P_g \subseteq B_f \cap B_g$. ■

A maximal fractional independent set *f* is said to be a *basic maximal fractional independent set* if there do not exist two maximal fractional independent sets *f* and *g*, and $\lambda \in (0, 1)$, such that $h = \lambda f + (1 - \lambda) g$. The characteristic function of a single vertex subset of $V(K_n)$ is a basic maximal fractional independent set of K_n, the complete graph of order *n*.

Theorem 1.12 [8]: A maximal fractional independent set *f* of *G* is basic if and only if there is no other maximal fractional independent set *g* such that $B_f = B_g$ and $P_f = P_g$. ■

A maximal fractional independent set *f* is said to be a *universal maximal fractional independent set*, if a convex combination of *f* with any other maximal fractional independent set is a maximal fractional independent set. The set of all universal maximal fractional independent sets is denoted by $\mathcal{F}_{UI}$.

Lemma 1.13 [8]: If *f* is a universal maximal fractional independent set of a graph *G*, then $B_f = V$. ■

Corollary 1.14 [8]: If *f* is a universal maximal fractional independent set of *G*, then

$$P_f \bigcap_g B_g ,$$

where the intersection is taken over all maximal fractional independent sets of *G*. ■

A characterization of graphs that have a universal maximal fractional independent set is provided in **[8]**.

Theorem 1.15 [8]: A graph *G* has a universal maximal fractional independent set if and only if there exists a unique partition of *V* into sets that induce maximal cliques. ■

The graphs that admit a universal maximal fractional independent set can be constructed from a disjoint union of *t* cliques (possibly of different sizes), $G_1, G_2, \ldots, G_t$. For each *i*, $1 \le i \le t$, select a non-empty subset $X_i \subseteq V(G_i)$ and any subset of edges joining vertices in

$$\bigcup_{i=1}^{t} \overline{X}_i .$$

The graph induced by these edges admits a universal maximal fractional independent set.

2. Minimal Fractional Perfect Dominating Sets and Fractional Efficient Dominating Sets

A set $S \subseteq V$ is called a *perfect dominating set* if, for every vertex $v \in V - S$, $|N(v) \cap S| = 1$. Yen and Lee **[9]** proved that, given a positive integer *k*, the problem of deciding if a graph *G* has a perfect dominating set of cardinality at most *k* is NP-complete. A *fractional perfect dominating set* of *G* is defined to be a function $f\colon V \to [0, 1]$ such that $f(N[v]) \ge 1$ if $f(v) > 0$ and $f(N[v]) = 1$ if $f(v) = 0$. A fractional perfect dominating set is a *minimal fractional perfect dominating set* if it is a fractional minimal dominating set.

Theorem 2.1: A fractional perfect dominating set *f* is minimal, if and only if $B_f \to P_f$. ■

A convex combination of two fractional perfect dominating sets *f* and *g*, $h_\lambda = \lambda f + (1 - \lambda) g$ is always a fractional perfect dominating set. Thus, the set of all fractional perfect dominating sets is a convex set. However, as the following theorem states, a convex combination of two minimal fractional perfect dominating sets may not be minimal. A minimal fractional perfect dominating set is a universal minimal fractional perfect dominating set if its convex combination with any other minimal fractional perfect dominating set is minimal. The condition for minimality of a convex combination is given in the following result, which shows that the set of all minimal fractional perfect dominating sets is not in general convex.

Theorem 2.2: A convex combination of two minimal fractional perfect dominating sets *f* and *g* is minimal, if and only if $B_f \cap B_g \to P_f \cup P_g$. ■

Theorem 2.3: If *f* is an minimal fractional perfect dominating set of *G*, then $B_f \to V(G)$.

Proof: Since *f* is minimal, $B_f \to P_f$. By definition of fractional perfect dominating set, if $f(v) = 0$, then $f(N[v]) = 1$. Hence, $B_f \to (V - P_f)$. ■

Theorem 2.4: If a graph *G* has an minimal fractional perfect dominating set *f* with $B_f = V(G)$, then *f* is a universal minimal fractional perfect dominating set.

Proof: Let *g* be any minimal fractional perfect dominating set of *G*. Then $B_g \to V$ and since $B_f = V(G)$, $B_f \cap B_g = V \cap B_g = B_g \to V$. Thus, the convex combination *f* and *g* is an minimal fractional perfect dominating set. Hence, *f* is a universal minimal fractional perfect dominating set. ■

A minimal fractional perfect dominating set of a graph *G* is a *basic minimal fractional perfect dominating set* if it cannot be expressed as a convex combination of two or more minimal fractional perfect dominating sets. The following results reveal the relationship between the set of all basic fractional minimal dominating sets and the set of all basic minimal fractional perfect dominating sets of a graph.

Theorem 2.5: If a basic fractional minimal dominating set f of a graph G is a minimal perfect dominating set, then it is a basic minimal fractional perfect dominating set.

Proof: Suppose, f is not a basic minimal fractional perfect dominating set. Then $f = \Sigma_i \lambda_i f_i$, where $\Sigma_i \lambda_i = 1$ and each f_i is a minimal fractional perfect dominating set. This shows that f is a convex combination of two or more fractional minimal dominating sets, which is a contradiction. ■

Theorem 2.6: If f is a basic minimal fractional perfect dominating set, then f is a basic fractional minimal dominating set.

Proof: Let f be a basic minimal fractional perfect dominating set and suppose that it is not a basic fractional minimal dominating set. Then there exist fractional minimal dominating sets $f_1, f_2, \ldots, f_r$ such that $f = \Sigma_i \lambda_i f_i$, where $\Sigma_i \lambda_i = 1$. The boundary and the positive set of f are $B_f = \cap_i B_{f_i}$ and $P_f = \cap_i P_{f_i}$, respectively. Thus, there exist infinitely many fractional minimal dominating sets whose boundary and positive sets are same as that of f. Hence, these fractional minimal dominating sets are minimal fractional perfect dominating sets. We can select two minimal fractional perfect dominating sets g_1 and g_2 such that $f = \lambda g_1 + (1 - \lambda) g_2$. Therefore, f can be expressed as a convex combination of at least two minimal fractional perfect dominating sets, a contradiction. ■

Lemma 2.7: If a minimal fractional perfect dominating set h is a convex combination of two fractional minimal dominating sets f and g and f is not an minimal fractional perfect dominating set, then there exists some $v \in V$ such that $f(v) = 0$ and $h(v) > 0$.

Proof: Since f is not a minimal fractional perfect dominating set, there exists $v \in V$ such that $f(v) = 0$ and $v \notin B_f$. Thus, $v \notin B_h$. Now suppose $h(v) = 0$. Since h is a minimal fractional perfect dominating set, v must be an element of B_h, which is a contradiction. ■

If, for a graph G, there exists a set of basic minimal fractional perfect dominating sets, say $B = \{f_1, f_2, \ldots, f_r\}$, such that every minimal fractional perfect dominating set of the graph is a convex combination of a subset of B, then the set of all minimal fractional perfect dominating sets is closed. The next result discusses the possibility of the set of all minimal fractional perfect dominating sets being closed.

Lemma 2.8: If there exists a minimal fractional perfect dominating set h, of a graph G such that if h can be expressed as a convex combination of a set of fractional minimal dominating sets that are not all minimal fractional perfect dominating sets, then the set of all minimal fractional perfect dominating sets of G is not a closed set.

Proof: Let the minimal fractional perfect dominating set h be a convex combination of a set of fractional minimal dominating sets say $\{f_1, f_2, \ldots, f_r\}$ such that f_1 is not a minimal fractional perfect dominating set. If $h = \Sigma_i \lambda_i f_i$, let $g = \Sigma_{i \geq 2} \lambda_i f_i$. The function g is a minimal fractional perfect dominating set. Define a sequence of real numbers (λ_{1j}) such that $\lambda_{1j} \to 1$. Let $h_j = \lambda_{1j} f_1 + (1 - \lambda_{1j}) g$. Since each h_j is a convex combinations of f_is, it is a minimal fractional perfect dominating set. Furthermore, $h_j \to f_1$ and f_1 is not in the set of all minimal fractional perfect dominating sets. Consequently, the set is not closed. ■

Next we discuss an example of a graph whose set of minimal fractional perfect dominating sets is open.

Example: Let $G = K_{1,n}$, where $n \geq 2$. Let the vertex set of G be $V(G) = \{v, v_1, v_2, \ldots, v_n\}$. Consider the functions f_1 and f_2 defined by $f_1(v) = 1$, $f_1(v_i) = 0$, $f_2(v) = 0$, and $f_2(v_i) = 1$ for $i = 1, 2, \ldots, n$. f_1 and f_2 are the only two basic fractional minimal dominating sets and all other fractional minimal dominating sets are convex combinations of these two basic fractional minimal dominating sets. Furthermore, all fractional minimal dominating sets except f_2 are minimal fractional dominating sets of G. Hence, the set of all minimal fractional perfect dominating sets of G is not closed.

Theorem 2.9: Let g be a fractional minimal dominating set of a graph G such that $w \in V(G)$ is g-sharp. Then g is not a universal minimal fractional perfect dominating set.

Proof: Let g be a fractional minimal dominating set of a graph G such that $w \in V(G)$ is g-sharp. Suppose that g is a universal minimal fractional perfect dominating set. Then, by definition of minimal fractional perfect dominating set, all $v \in V$ with $g(v) = 0$ must be in B_g. Thus, by Lemma 1.4, $g(w) = 0$ and $w \notin B_g$, which is a contradiction. ■

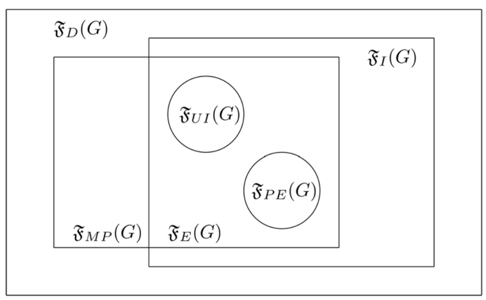
According to the definition given by Bang *et al.* **[10][11]**, a dominating set S of G is called an *efficient dominating set* of G if the closed neighborhood of every vertex in V has exactly one vertex of S. Equivalently, S is an efficient dominating set if S dominates G and the distance between any two elements of S is at least three. Grinstead and Slater **[12]** introduced the idea of fractional efficient domination. A *fractional efficient dominating set* of a graph $G = (V, E)$ is a dominating function $f: V \to [0, 1]$ such that $\Sigma_{x \in N[v]} f(x) = 1$ for all $v \in V$. If f is a fractional efficient dominating set, then $B_f = V \to P_f$. Thus, every fractional efficient dominating set of a graph G is a fractional minimal dominating set. Grinstead and Slater **[12]** also proved that, if f is a fractional efficient dominating set of G, then $\gamma(G) = |f| = \Sigma_{x \in V} f(x)$. It is interesting to note that the conditions for minimal fractional perfect dominating set and the conditions for maximal fractional independent set together give the conditions for fractional efficient dominating set. A fractional efficient dominating set f of G is a *positive fractional efficient dominating set* if $P_f = V$. For convenience, denote the set of all fractional efficient dominating sets of G by $\mathcal{F}_E(G)$. Additionally, let $\mathcal{F}_{PE}(G)$ and $\mathcal{F}_{MP}(G)$ denote the set of all minimal fractional perfect dominating sets and the set of all positive fractional efficient dominating sets, respectively. Then $\mathcal{F}_{PE}(G) \cap \mathcal{F}_{MP}(G) = \mathcal{F}_E(G)$. We note the following:

Observation 2.10: The set of all positive fractional efficient dominating sets and the set of all maximal fractional independent sets are subsets of the set of all fractional minimal dominating sets.

Observation 2.11: If a function is both a positive fractional efficient dominating set and a maximal fractional independent set, then it is a fractional efficient dominating set.

Observation 2.12: If f is a universal maximal fractional independent set of G, then f is a fractional efficient dominating set.

These observations are summarized in the following figure.



Figure

A convex combination of two fractional efficient dominating sets f and g is minimal if and only if $B_f \cap B_g \to P_f \cup P_g$. Naturally, there exists a fractional efficient dominating set that cannot be expressed as a convex combination of two or more fractional efficient dominating sets. Such functions are basic fractional efficient dominating sets. The next result characterizes basic fractional efficient dominating sets of a graph.

> **Theorem 2.13:** A fractional efficient dominating set f of a graph G is a basic fractional efficient dominating set if and only if there exists no other fractional efficient dominating set g such that $P_f = P_g$.

Proof: Let f be a basic fractional efficient dominating set and suppose there exists an fractional efficient dominating set g with the property $P_f = P_g$. Because f and g are maximal fractional independent sets, then by Theorem 1.12, f is not a basic maximal fractional independent set. Consequently, there exists a set of maximal fractional independent sets $\{f_1, f_2, \ldots, f_r\}$ such that $f = \Sigma_i \lambda_i f_i$, $B_f = \cap_i B_{f_i}$, and $P_f = \cup_i P_{f_i}$. Since $B_f = V$, then $B_{f_i} = V$ for all i, and hence, $P_{f_i} \subseteq B_{f_i}$ for all i. Thus, each f_i is a fractional efficient dominating set. This contradicts the fact that f is a basic fractional efficient dominating set.

Conversely, suppose there is no fractional efficient dominating set g such that $B_f = B_g$ and $P_f = P_g$, and that f is not a basic fractional efficient dominating set. Then, by definition of basic fractional efficient dominating set, there exists a set of fractional efficient dominating sets $\{f_1, f_2, \ldots, f_r\}$ such that $f = h_{\lambda_1} = \Sigma_i \lambda_i f_i$, where $\Sigma_i \lambda_i = 1$ and $0 < \lambda_i < 1$. Furthermore, $B_f = \cap_i B_{f_i}$ and $P_f = \cup_i P_{f_i}$. Since Now for $\lambda_{11} \neq \lambda_{12}$, $h_{\lambda_{11}}$ and $h_{\lambda_{12}}$ are fractional efficient dominating sets with $B_{h_{\lambda_{1i}}} = B_f$ and $P_{h_{\lambda_{1i}}} = P_f$. This is a contradiction.

■

Theorem 2.14: Every fractional efficient dominating set of G is a maximal fractional independent set.

Proof: If f is a fractional efficient dominating set of G, then $B_f = V$ and $P_f \subseteq V$. Thus, $P_f \subseteq B_f$, and hence, f is an maximal fractional independent set of G. ■

A convex combination of two fractional efficient dominating sets (positive fractional efficient dominating sets) f and g of a graph G is a function $h = \lambda f + (1-\lambda)g$, where $0 < \lambda < 1$. A positive fractional efficient dominating set f is a *basic positive fractional efficient dominating set* if it is not a convex combination of two or more positive fractional efficient dominating sets. Let f be a fractional efficient dominating set of G. We define

$$C_0(f) = \{v \in V : f(v) = 0\} \text{ and } C_1(f) = \{v \in V : f(v) = 1\}.$$

Lemma 2.15: If f is a positive fractional efficient dominating set of G, then

1. $C_0(f) = \varnothing$, and
2. $C_1(f) = \varnothing$.

Proof: The first statement follows from the definition of positive fractional efficient dominating set.

To prove Statement 2, suppose that f is a positive fractional efficient dominating set of G and $x \in C_1(f)$. Because f is a fractional minimal dominating set, $(B_f \cap N(x)) \neq \varnothing$. Let $y \in B_f \cap N(x)$. Since $|N(y)| \geq 2$ and $x \in N(y)$, there exists $z \in N(y)$ such that $f(z) = 0$. This contradicts the premise $C_0(f) = \varnothing$. ■

Theorem 2.16: If a graph G has a basic positive fractional efficient dominating set, then G has no other positive fractional efficient dominating set.

Proof: Suppose a graph G has a basic positive fractional efficient dominating set f and another positive fractional efficient dominating set g. Since $B_f = B_g = V$ and $P_f = P_g = V$, the systems of equations corresponding to f and g are the same. Thus, by Theorem 1.7, we obtain a system of equations that has simultaneously a unique solution and infinitely many solutions, a contradiction. ■

Corollary 2.17: If a graph G has two positive fractional efficient dominating sets, then G has no basic positive fractional efficient dominating set. ■

Theorem 2.18: The set of all fractional efficient dominating sets and the set of all positive fractional efficient dominating sets of a graph G are convex.

Proof: Let f and g be two fractional efficient dominating sets of a graph G and let $h = \lambda f + (1-\lambda)g$ be their convex combination. Then, $B_h = B_f \cap B_g = V$. Consequently, h is an fractional efficient dominating set.

Similarly, let f and g be two positive fractional efficient dominating sets of G and let h be their convex combination. Then, h is a fractional efficient dominating set. Furthermore, $P_h = P_f \cup P_g = V$. Hence, h is a positive fractional efficient dominating set. ■

Theorem 2.19: Let G be a graph such that $\mathcal{F}_E(G) \neq \varnothing$. Then $\mathcal{F}_E(G)$ is a closed subset of $\mathcal{F}_D(G)$.

Proof: By Theorem 2.18, $\mathcal{F}_E(G)$ is a convex set. First, we show that if a fractional efficient dominating set f of G is a convex combination of a set of fractional minimal dominating sets, say $\{f_1, f_2, \ldots, f_r\}$, then f_i is a fractional efficient dominating set for all i.

We know that $V = B_f = \cap_i B_{f_i} \subseteq B_{f_i}$ for all i. Thus, $B_{f_i} = V$ for all i, and hence, each f_i is a fractional efficient dominating set. The rest of the proof follows easily. ■

Now we prove that if $\mathcal{F}_{PE}(G)$ is not a singleton set, then $\mathcal{F}_{PE}(G)$ is an open set. For this we need the following definitions from **[6]**. Let f and g be two fractional minimal dominating sets of a graph G, with $B_f = B_g$ and $P_f = P_g$. Let i be any real number and let $h_i = (1+i)g - if$. Clearly $h_0 = g$ and $h_{-1} = f$. Now, let S be the subset of the set of real numbers $\mathbb{R}$ defined by

$$S = \{i : i \in \mathbb{R}, h_i \text{ is a fractional dominating set such that } B_{h_i} = B_g \text{ and } P_{h_i} = P_g\}.$$

In general, h_i is not a fractional dominating set because h_i becomes negative for large values of i.

Lemma 2.20 [6]: If $i \in S$, then h_i is a fractional minimal dominating set of *G*. ■

Lemma 2.21 [6]: *S* is a bounded open interval of $\mathbb{R}$. ■

It may be noted that for any h_i, where $i \in S$ is a convex combination of two fractional minimal dominating sets h_k and h_l, where $l, k \in S$, the set $\{h_i : i \in S\}$ is an open set.

Theorem 2.22: Let **G** be a graph such that $\mathcal{F}_{PE}(G)$ contains at least two elements. Then, $\mathcal{F}_{PE}(G)$ is an open subset of $\mathcal{F}_D(G)$.

Proof: First, we claim that if *f* and *g* are two partial fractional efficient dominating sets of a graph *G*, then any $h_i = (1+i)g - if$, $i \in S$, is a positive fractional efficient dominating set. For all $i \in S$, h_i is a fractional minimal dominating set. Additionally, $B_{h_i} = B_g = V$ and $P_{h_i} = P_g = V$. Thus, h_i is a positive fractional efficient dominating set. Now, by the remark given above, any h_i, where $i \in S$, is a convex combination of two positive fractional efficient dominating sets h_k and h_l, where $l, k \in S$. Hence, for every positive fractional efficient dominating set *f*, there exists an open subset of $\mathcal{F}_{PE}(G)$. ■

Proposition 2.23: If a graph *G* contains two vertices *u* and *v* such that $N[u] \subset N[v]$, then $f(x) = 0$ for all $x \in (N[v] - N[u])$, where *f* is any fractional efficient dominating set of *G*.

Proof: Let *f* be a fractional efficient dominating set of *G*. Then, by definition of fractional efficient dominating set, $f(N[v]) = f(N[u]) = 1$. Also, $f(N[v]) = f(N[u]) + \Sigma_{x \in (N[v]-N[u])} f(x) = 1$. Hence, $f(x) = 0$ for all $x \in (N[v] - N[u])$. ■

Corollary 2.24: If a graph *G* contains two vertices *u* and *v* such that $N[u] \subset N[v]$, then *G* has no positive fractional efficient dominating set *f* with $B_f = P_f = V$. ■

Theorem 2.25: A positive fractional efficient dominating set *f* of a graph *G* is a universal maximal fractional independent set, if and only if the set of all maximal fractional independent sets and the set of all fractional efficient dominating sets are equal.

Proof: Suppose that a positive fractional efficient dominating set *f* of *G* is a universal maximal fractional independent set. Then, by Corollary 1.14, $P_f = V \subseteq \cap_g B_g$, where the intersection is taken over all maximal fractional independent sets of *G*. Thus, $B_g = V$ for any maximal fractional independent set, *g*, of *G*. Hence, *g* is an fractional efficient dominating set.

Conversely, suppose that any maximal fractional independent set, *g*, is an fractional efficient dominating set of *G*. Then, $B_g = V$. If *f* and *g* are two different maximal fractional independent sets of *G*, then $P_f \cup P_g \subseteq B_f \cap B_g = V$. Thus, any convex combination of *f* and *g* is a maximal fractional independent set. Consequently, $\mathcal{F}_I(G)$ is a convex set.

In other words, the set of all maximal fractional independent sets and the set of all universal maximal fractional independent sets are equal. Since, $\mathcal{F}_{PE}(G) \subseteq \mathcal{F}_I(G)$, it follows that $\mathcal{F}_{PE}(G) \subseteq \mathcal{F}_{UI}(G)$. ■

Theorem 2.26: Every fractional efficient dominating set of a graph *G* is a universal minimal fractional perfect dominating set of *G*.

Proof: Let *f* be a fractional efficient dominating set of *G*. Then $B_f = V$. Let *g* be any minimal fractional perfect dominating set of *G*. Then B_g dominates *V* and $V - P_g \subseteq B_g$. Consider *h*, a convex combination of *f* and *g*. Then $B_h = B_f \cap B_g = B_g$ and B_g dominates *V*. Thus, *h* is a fractional minimal dominating set. Since $V - P_h = (V - P_g) \cap (V - P_f)$ and $(V - P_h) \subseteq (V - P_g) \subseteq B_g = B_h$, *h* is a minimal fractional perfect dominating set. This shows that *f* is a universal minimal fractional perfect dominating set. ■

There are many classes of graphs having at least one fractional efficient dominating set. The following construction is useful for a class of graphs that have no fractional efficient dominating set. Let *H* be a graph containing a vertex *v* such that there exist three vertices u_1, u_2, and *w* such that $|N[u_i]| \geq 2$, $N[u_i] \subseteq N[v]$, for $i = 1, 2$, and $N[u_1] \cap N[u_2] = \{v\}$, and $N(w) \subseteq N(v)$. Let *F* be the class of all graphs having the above property. In the next theorem, we prove that the graphs in *F* have no fractional efficient dominating set.

Theorem 2.27: If $G \in F$, then G has no fractional efficient dominating set.

Proof: Let $G \in F$ and let $v \in V(G)$ be a vertex such that there exist vertices u_1, u_2, and w satisfying the conditions given above. Suppose that f is a fractional efficient dominating set of G. We claim that $f(v) = 1$. By definition of F, $N[u_1] \cap N[u_2] = \{v\}$ and $|N[u_1]| \geq 2$. If $f(v) < 1$, then $f(x) > 0$ for some $x \in (N[v_i] - \{v\})$, where $i = 1, 2$. Thus, $f(N[v]) = f(N[u_1]) + f(N[u_2]) - f(v) > 1$, which contradicts the fact that f is a fractional efficient dominating set. Consequently, $f(x) = 0$ for all $x \in v(\ )$ and since $f(N[w]) = 1$, $f(w) = 1$. Because there is a vertex $x \in N(v)$ that is adjacent to both v and w, then $f(N[x]) > 1$ for some $x \in N(v)$, a contradiction. ■

The classes of graphs that admit universal maximal fractional independent sets are known. By Theorem 2.25, the set of all graphs G, such that $\mathcal{F}_E(G) = \mathcal{F}_I(G)$ must be a subset of this class.

Theorem 2.28: Let G be a graph. Then $\mathcal{F}_E(G) = \mathcal{F}_I(G)$ if and only if G is a disjoint union of t disjoint cliques $G_1, G_2, \ldots, G_t$ (possibly of different orders).

Proof: By Theorem 1.15 a graph G has a universal maximal fractional independent set if and only if there exists a unique partition of V into sets that induces maximal cliques. If G belongs to this class and is not the disjoint union of t disjoint cliques, there exist two vertices u and v such that $N[u] \subset N[v]$. Thus, by Theorem 2.27, G cannot have any positive fractional efficient dominating set. Hence, $\mathcal{F}_E(G) = \mathcal{F}_I(G)$. Proof of the converse is straightforward. ■

3. Conclusion and Open Questions

Theorem 2.9 gives a sufficient condition for the set of all minimal fractional perfect dominating sets to be a proper subset of the set of all minimal fractional dominating sets. Whether this condition is necessary, is an interesting question for further investigation. If a universal minimal fractional dominating set f is a minimal fractional perfect dominating set, then f is a universal minimal fractional perfect dominating set. Whether any graph has a universal minimal fractional perfect dominating set that is not a universal minimal fractional dominating set is still unknown. If a graph G has a universal minimal fractional perfect dominating set does it guarantee the existence of a positive fractional efficient dominating set? Similarly, does the existence of a positive fractional efficient dominating set imply the existence of a universal minimal fractional perfect dominating set? Having characterized the graphs such that $\mathcal{F}_E(G) = \mathcal{F}_I(G)$ (see Theorem 2.28) is it possible to characterize those graphs having at least one fractional efficient dominating set as well as those graphs having no fractional minimal dominating set other than fractional efficient dominating set?

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(1) Short contributions, surveys, and research articles in graph theory, related fields, and applications.

(2) Graph theory problems (clean or dirty), partial solutions to problems.

(3) Information about availability of graph theory information such as, preprints and technical reports.

(4) Developments on articles in previous *Graph Theory Notes of New York.*

Graph Theory Notes of New York is intended to provide for informal exchange of ideas within the graph theory community. Thus, a good selection of references should be included in each submission.

Please submit contributions to the Editors following the format of a recent issue.

All manuscripts submitted to *Graph Theory Notes of New York* are refereed. Acceptance is the responsibility of the Editors. Accepted articles will not necessarily be published in the order they are received but with the objective of balancing the content of each issue with respect to diversity of topics and authors.

Preferred articles should be concise, self-contained with respect to definitions, provide adequate background to the subject matter. In general, proofs should be in outline form rather than detailed. Papers containing open problems or partial solutions are especially welcome.

Excessive use of figures is discouraged. If used, art work must be submitted in camera ready form (good quality ink drawings or 1200 dpi laser printer output are acceptable). Text used as items contained in the art work should be in Times Roman and variables used should be italicized. Please note that figures may be subjected to considerable size reduction and this should be allowed for when preparing the originals.

Electronic submissions are welcome but must be accompanied by a printed copy of the manuscript.

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