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Editors:
Daniel Gagliardi
John W. Kennedy
Louis V. Quintas

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CONTENTS

INTRODUCTORY REMARKS

Here we are at *Graph Theory Notes of New York*, Issue LXII. We would like to welcome back old friends and invite new ones to enjoy another issue of our journal. Our debut issue was published in 1980 and with this one we are officially thirty two years young! We are indeed most grateful to all of those authors who have made contributions to *Graph Theory Notes* over the years. We would also like to give a special thanks to the referees, associate editors, and to others who have spent countless hours reviewing submissions. Your help has been indispensable.

As it is with every other aspect of human activity, we at *Graph Theory Notes* continue to evolve with the rapid pace of advances in technology. For example, we now have regular editorial meetings through video chat and are able to check citations from the comfort of our homes. One of the recent editorial changes we have made relates to the submission process for authors. We now accept articles in electronic form. The preferred format is a text-based pdf file. We will only ask for a printed copy of a submitted article to resolve ambiguities in the text or artwork. More information concerning submission of manuscripts is provided in the Instructions for Contributors inside the back cover of this issue.

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and send to the editorial address shown inside the front cover.

As many of you know, *Graph Theory Notes*, along with the Metropolitan Section of the Mathematical Association of America sponsors a biannual one day conference for enthusiasts to present their current work and exchange ideas in graph theory. These **Graph Theory Days** are usually held at institutions of higher learning in the greater New York area. We welcome all to attend Graph Theory Days and especially to invite students to learn more about this subject which we deeply love. We also strongly encourage people to consider arranging for their institution to host a Graph Theory Day. Such support, although non-trivial, is not difficult to effect — especially with our help. Moreover, the satisfaction and positive publicity for hosts and their institutions is well worth the very modest financial investment.

We close with an announcement about our website. Thanks to Mike Kazlow's efforts those interested in *Graph Theory Notes* and Graph Theory Day can get information and past issues by going to **<http://gtn.kazlow.info/>**

The editors are energized by the support we have received over the years from our colleagues and readers. Your enthusiasm plays a pivotal role in inspiring further study in mathematics and in developing an appreciation for its beauty.

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August 2012

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[GTN LXII:1] PARTIAL PROOFS FOR REED'S CONJECTURE

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1. Reed's Conjecture

Reed's Conjecture is an extension of a well-known upper bound for the *chromatic number* $\chi(G)$ of a graph G. **[1]**.

Conjecture (Reed): For any graph G

(1) $$\chi(G) \le \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil,$$

where $\Delta(G)$ denotes the maximum degree and $\omega(G)$ is the clique number of G. □

Odd cycles of length at least 5, and the 4-regular, 4-chromatic, triangle-free graph devised by Chvátal **[2]** all show that the ceiling in **(1)** is necessary.

Reed's Conjecture was first published in 1998 **[3]** and at this time no proof (or counterexample) is known. Because there is a multitude of partial proofs, it seems to be worthwhile to publish an ordered compilation of these partial results together with references.

2. Notation

In this paper *graph coloring* refers to the vertex-coloration of graphs. All graphs considered are simple (finite, undirected, with no loop or multiple edge). As usual, we suppose G to be a graph with p vertices and q edges, and that $\overline{G}$ is the complement of G. Special graph classes are the complete graphs K_p, the cycle graphs C_p, and the claw $K_{1,3}$. Frequently occurring variables are the *minimum degree* δ and the *maximum degree* Δ, the *clique number* ω (the number of vertices of the largest clique), the *independence number* β_0, and the *vertex-cover number* α_0. We write χ instead of $\chi(G)$ and so forth, when the context is clear.

3. Principles Underlying the Compilation of Partial Proofs

The list that follows was compiled from a variety of printed sources so that, for any entry, results can be traced back to their origin. In each case the *most powerful* statement was selected (some historical material can be found in **[4]**). This principle was partially relaxed for the sake of easy reading (the entry *planar and toroidal graphs* contains some redundancy).

The *Main List* (Section 4.1) addresses classes of graphs. For example, *line graphs* means that **(1)** has been verified for all line graphs. The list also contains inequalities; for example, $\chi(G) \le \omega(G) + 2$ means that any graph G obeying this inequality also satisfies **(1)**.

One suggestion for practical work that cannot be represented in this style is that we can restrict our study to *color-critical graphs* with *chromatic number* χ (χ-critical graphs). Every graph with chromatic number χ contains a χ-critical subgraph with the same number of vertices. If such a graph satisfies **(1)**, then for any other graph generated from it by inserting edges, Δ and ω remain constant or increase, so that **(1)** will remain valid. Consequently, we can make use of the known properties of color-critical graphs. Also for practical purposes it is useful sometimes to consider lower bounds for graph invariants of a hypothetical counterexample (provided this made clear in the text).

4. Compilation of Partial Solutions

4.1. Main List

We start with classes of graphs that can be characterized by specific terms.

1. Planar and toroidal graphs. **[4]**
2. Decomposable graphs (graphs with disconnected complements). **[5]**
3. Line graphs, generalized line graphs, and exceptional graphs with $\lambda_p > -2$, where λ_p is the smallest eigenvalue of the adjacency matrix of the graph. **[4]**
4. Claw-free graphs (that is, $K_{1,3}$-free graphs). **[6]**
5. Quasi-line graphs. **[7][8]**
6. Almost-split graphs. **[9]**

Reed's Conjecture has been verified for special classes of graphs that are defined by forbidden induced subgraphs. (The following list is not complete.)

7. Odd-hole free graphs. **[7]**
8. $3K_1$-free graphs (graphs G with $\beta_0 \le 2$). **[6]**
9. $\{2K_2, C_4\}$-free graphs. **[4]**
10. $\{P_5, C_4\}$-free graphs. **[7]**
11. Every {even-hole, diamond}-free graph satisfies $\chi \le \omega + 1$ **[10]**. Hence, the Conjecture holds (see item 14) for these graphs. Every counterexample to Reed's Conjecture has an even hole and/or a diamond as an induced subgraph.
12. Aravind, *et al*. **[7]** present many theorems stating that Reed's Conjecture holds for classes of graphs defined by families of forbidden induced subgraphs (for example, all {chair, house, bull, dart}-free graphs).

The following results have the form of an inequality (whenever at least one of these inequalities holds for a graph G, then **(1)** is also valid for G).

13. $\beta_0 \le 2$. **[6]**
14. $\chi \le \omega + 2$. **[4]**
15. $\Delta \ge p - 7$. **[9]**
16. $\Delta \ge p - \beta_0 - 4$. **[9]**
17. $\Delta \ge p + 2 - (\beta_0 + \sqrt{p + 5 - \beta_0})$. **[11]**
18. $\chi > \lceil p/2 \rceil$. **[9][11]**
19. $\chi > (p - \beta_0 + 3)/2$. **[9][11]**

4.2. Inequalities Derived Using the Theory of Graph Associations

A theorem obtained using the concept of *graph associations* **[5]** allows us to derive new upper bounds for χ by choosing special types of induced subgraphs.

Definition: Given a graph G and non-adjacent vertices a and b, we write $G/[a, b]$ for the graph obtained from G by associating (that is, identifying) a and b into a single vertex $[a, b]$ and discarding any multiple edges. ■

Theorem: Let G be a graph of order $p(G)$. Then, for any induced subgraph H G

(2) $$\chi(G) \le \chi(H) + \frac{p(G) + \omega(G) - p(H) - 1}{2}$$, where $p(H)$ is the order of H. ■

One immediate application of this result is given as an example. If G is a connected graph, and if the subgraph H is identified with a longest induced path P_m of G ($m \ge 3$, such that the diameter $d(P_m) = d(G) = m - 1 \ge 2$), then **(2)** leads to

(3) $$\chi(G) \le \frac{p(G) + \omega(G) - d(G) + 2}{2}.$$

Operations like this make it possible to make use of additional known graph properties. This is an extremely condensed version, for more details see **[4][5]**.

4.3. Miscellaneous Remarks

1. A lower bound for the number of edges: For color-critical graphs of order p and chromatic number χ a good lower bound for the number of edges q is still required. At present, the best such bound is due to Kostochka, *et al*. **[12]**, under the reservation that three classes of exceptional graphs are excluded. This restriction is the conjunction of three properties, best expressed in the following statement:

(4) If G is a color-critical graph and $4 \le \chi \le p-2$ and
$(2\chi \ne p+1$ or $\beta_0 \ge 3$ or $\omega < (p-1)/2)$, then $q \ge p(\chi-1)/2 + \chi - 3$.

In our work, **(4)** always can be applied, since $\beta_0 \ge 3$ is guaranteed for counterexamples to **(1)**.

2. A list of lower bounds: In any counterexample to **(1)**, the variables must satisfy the following (small selection of) lower bounds:

$$p \ge 12\ ,\ q \ge 34\ ,\ \chi \ge 5\ ,\ \delta \ge 4\ ,\ \Delta \ge 5\ ,\ \beta_0 \ge 3\ ,\ \alpha_0 \ge 8\ ,\text{ and }\gamma \ge 2\ .$$

Additional details and necessary properties for the complement $\overline{G}$ of G can be found in **[4][5]**.

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[GTN LXII:2] EDGE DEGREE WEIGHT SUM OF MYCIELSKIAN GRAPHS AND PRODUCT GRAPHS

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Abstract

Let $G = (V, E)$ be a graph. For each edge $e = uv$ in G, a weight $w(e) = \deg(u) + \deg(v)$ is assigned. The edge-degree sum of G is defined to be the sum of the edge weights of G. In this paper, we present results for Mycielskian graphs and the four standard products of graphs.

1. Introduction

Let $G = (V, E)$ be a graph of order p and size q. Among the variety of *labeling* of a graph **[1]**, a type of weighted edge labeling using the sum of the degrees of its adjacent vertices, was introduced in **[2]**. For each edge $e = uv \in E(G)$, the weight $w(e)$ of e is defined by $w(e) = \deg_G(u) + \deg_G(v)$, where $\deg_G(v)$ is the degree of vertex v in G. The *edge degree weighted sum* $w(G)$ of G is defined by

$$w(G) = \sum_{e \in E(G)} w(e) .$$

Theorem 1.1 [2]: For any graph G, $w(G) = \sum_{v \in V(G)} (\deg_G(v))^2$ ■

In **[3]**, the general problem is posed to obtain $w(G)$ for a class of graphs G. Here we do this for *Mycielskian graphs* and for the four standard products of graphs; namely, the *Cartesian product*, the *strong product*, the *direct product*, and the *lexicographic product*. For additional information on product graphs see **[4][5].**

2. Mycielskian Graphs

The *Mycielskian graph* $\mu(G)$ of $G = (V, E)$ is defined as follows: If $V = \{v_1, v_2, \ldots, v_p\}$, then $V(\mu(G)) = V \cup V' \cup \{x\}$, where $V' = \{u_1, u_2, \ldots, u_p\}$ and

$$E(\mu(G)) = E(G) \cup \{u_i v_j : v_i v_j \in E(G)\} \cup \{x u_i : 1 \le i \le p\} .$$

Define $\mu_k(G) = \mu(\mu_{k-1}(G))$, $k \ge 1$, and $\mu_0(G) = G$.

Lemma 2.1: Let G be a graph. Then

(1) $|V(\mu_k(G))| = 2^k(|V(G)| + 1) - 1$.

(2) $|E(\mu_k(G))| = 3^k|E(G)| + (3^k - 2^k)(|V(G)| + 1) - \left(\frac{3^k - 1}{2}\right)$.

Proof: We prove these results by induction on k.

(1) For $k = 1$, from the definition of $\mu_1(G)$, $|V(\mu_1(G))| = 2|V(G)| + 1$. Therefore, statement (1) is true for $k = 1$. Assume that statement (1) is true for some $k - 1 \ge 0$. From the definition $\mu_k(G) = \mu(\mu_{k-1}(G))$ and by the induction hypothesis,

$$\begin{aligned} |V(\mu_k(G))| &= 2|V(\mu_{k-1}(G))| + 1 \\ &= 2[2^{k-1}(|V(G)| + 1) - 1] + 1 \\ &= 2^k(|V(G)| + 1) - 1. \end{aligned}$$

(2) For $k = 1$, from the definition of $\mu_1(G)$, $|E(\mu_1(G))| = 3|E(G)| + |V(G)|$. Therefore, statement (2) is true for $k = 1$. Assume that statement (2) is true for $k-1$. Then from the definition of $\mu_k(G)$, statement (1), and the induction hypothesis,

$$\begin{aligned}|E(\mu_k(G))| &= 3|E(\mu_{k-1}(G))| + |V(\mu_{k-1}(G))| \\ &= 3\left[3^{k-1}|E(G)| + (3^{k-1} - 2^{k-1})(|V(G)| + 1) - \left(\frac{3^{k-1} - 1}{2}\right)\right] + 2^{k-1}(|V(G)| + 1) - 1 \\ &= 3^k|E(G)| + (3^k - 2^k)(|V(G)| + 1) - \left(\frac{3^k - 1}{2}\right).\end{aligned}$$

Thus, statements (1) and (2) are true for any integer $k \geq 1$. ■

Lemma 2.2: Let $G = (V, E)$ be a graph. Then $w(\mu(G)) = 5w(G) + 4|E(G)| + |V(G)|(|V(G)| + 1)$.

Proof: By Theorem 1.1,

$$\begin{aligned}w(\mu(G)) &= \sum_{v \in V(\mu(G))} (\deg(v))^2 \\ &= \sum_{v \in V} (\deg_{\mu(G)}(v))^2 + \sum_{u \in V'} (\deg_{\mu(G)}(u))^2 + (\deg_{\mu(G)}(z))^2 \\ &= \sum_{v \in V} (2\deg_G(v))^2 + \sum_{v \in V} (\deg_G(v) + 1)^2 + (|V(G)|)^2 \\ &= 5w(G) + 2\sum_{v \in V} \deg_G(v) + |V(G)|(|V(G)| + 1) \\ &= 5w(G) + \mu_k(G) + |V(G)|(|V(G)| + 1).\end{aligned}$$

■

Theorem 2.3: Let G be a graph. Then

$$\begin{aligned}w(\mu_k(G)) &= 5^k w(G) + 2(5^k - 3^k)|E(G)| + (5^k - 4^k)(|V(G)| + 1)^2 \\ &\quad + \frac{1}{3}(5^k - 2 \cdot 3^{k+1} + 5 \cdot 2^k)(|V(G)| + 1) - \frac{1}{2}(5^k - 2 \cdot 3^k + 1).\end{aligned}$$

Proof: We prove this theorem by induction on k.

By Lemma 2.2, the result is true for $k = 1$. Assume that the result is true for $k-1$. By Lemma 2.2,

$$\begin{aligned}w(\mu_k(G)) &= 5w(\mu_{k-1}(G)) + 4|E(\mu_{k-1}(G))| + |V(\mu_{k-1}(G))|(|V(\mu_{k-1}(G))| + 1) \\ &= 5[5^{k-1}w(G) + 2(5^{k-1} - 3^{k-1})|E(G)| + (5^{k-1} - 4^{k-1})(|V(G)| + 1)^2 \\ &\quad + \frac{1}{3}(5^{k-1} - 2 \cdot 3^k + 5 \cdot 2^{k-1})(|V(G)| + 1) - \frac{1}{2}(5^{k-1} - 2 \cdot 3^{k-1} + 1)] \\ &\quad + 4\left[3^{k-1}|E(G)| + (3^{k-1} - 2^{k-1})(|V(G)| + 1) - \frac{3^{k-1} - 1}{2}\right] \\ &\quad + [2^{k-1}(|V(G)| + 1) - 1]2^{k-1}(|V(G)| + 1).\end{aligned}$$

Hence, using the induction hypothesis and Lemma 2.1,

$$
\begin{aligned}
w(\mu_k(G)) &= 5^k w(G) + [2(5^k - 5\cdot 3^{k-1}) + 4\cdot 3^{k-1}]|E(G)| + (5^k - 5\cdot 4^{k-1} + 2^{2k-2})(|V(G)|+1)^2 \\
&\quad + \left[\frac{5}{3}(5^{k-1} - 2\cdot 3^k + 5\cdot 2^{k-1}) + 4\cdot 3^{k-1} - 5\cdot 2^{k-1}\right](|V(G)|+1) \\
&\quad - \frac{5}{2}(5^{k-1} - 2\cdot 3^{k-1} + 1) - 2(3^{k-1} - 1) \\
&= 5^k w(G) + 2(5^k - 3^k)|E(G)| + (5^k - 4^k)(|V(G)|+1)^2 \\
&\quad + \frac{1}{3}(5^k - 2\cdot 3^{k+1} + 5\cdot 2^k)(|V(G)|+1) - \frac{1}{2}(5^k - 2\cdot 3^k + 1).
\end{aligned}
$$

Thus, the theorem is true for any integer $k \geq 1$. ■

3. Graph Products

The four standard products of graphs are the *Cartesian product*, the *strong product*, the *direct product*, and the *lexicographic product*. Results concerning the Cartesian product are given in **[3]**. The weight for the other three products are derived in what follows. The vertex sets of each of the four products of two graphs is the Cartesian product of the vertex sets of the factor graphs. In the *strong product* $G \otimes H$ of two graphs G and H, two vertices (u, v) and (x, y) in $V(G \otimes H)$ are adjacent if and only if $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$ or $ux \in E(G)$ and $vy \in E(H)$.

Two vertices (u, v) and (x, y) in the *direct product* $G \times H$ are adjacent if and only if $ux \in E(G)$ and $vy \in E(H)$.

Two vertices (u, v) and (x, y) in the *lexicographic product* $G \circ H$ are adjacent if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$.

Theorem 3.1 [3]: Let G and H be disjoint graphs of order p_1 and p_2, and size q_1 and q_2, respectively. Let $G \,\square\, H$ denote the Cartesian product of G and H. Then
$w(G \,\square\, H) = p_2 w(G) + p_1 w(H) + 8q_1q_2$. ■

Theorem 3.2: Let G and H be disjoint graphs of order p_1 and p_2, and size q_1 and q_2, respectively. Let $G \otimes H$ denote the strong product of G and H. Then,
$w(G \otimes H) = w(G)(p_2 + 4q_2) + w(H)(p_1 + 4q_1) + w(G)w(H) + 8q_1q_2$.

Proof: Let $(u, v) \in V(G \otimes H)$. Then

$$
\begin{aligned}
w(G \otimes H) &= \sum_{(u,v) \in V(G \otimes H)} (\deg_{G \otimes H}(u, v))^2 \\
&= \sum_{(u,v) \in V(G \otimes H)} (\deg_G(u) + \deg_H(v) + \deg_G(u)\deg_H(v))^2 \\
&= \sum_{(u,v) \in V(G \otimes H)} \{(\deg_G(u))^2 + (\deg_H(v))^2 + (\deg_H(v))^2(\deg_H(v))^2 \\
&\qquad + 2[\deg_G(u)\deg_H(v) + (\deg_G(u))^2\deg_H(v) + \deg_G(u)(\deg_H(v))^2]\} \\
&= p_2 \sum_{u \in V(G)} (\deg_G(u))^2 + p_1 \sum_{v \in V(H)} (\deg_H(v))^2 \\
&\quad + 2\Bigg[\sum_{u \in V(G)} \deg_G(u) \sum_{v \in V(H)} \deg_H(v) \\
&\quad + \sum_{u \in V(G)} (\deg_G(u))^2 \sum_{v \in V(H)} \deg_H(v) + \sum_{u \in V(G)} \deg_G(u) \sum_{v \in V(H)} (\deg_H(v))^2\Bigg]
\end{aligned}
$$

Hence,

$$\begin{aligned} w(G \otimes H) &= p_2 w(G) + p_1 w(H) + w(G)w(H) + 8q_1q_2 + 4q_2 w(G) + 4q_1 w(H) \\ &= w(G)(p_2 + 4q_2) + w(H)(p_1 + 4q_1) + w(G)w(H) + 8q_1q_2. \end{aligned}$$

■

Corollary 3.3: Let G and H be disjoint regular graphs of order p_1 and p_2, and degree r_1 and r_2, respectively. Then $w(G \otimes H) = p_1p_2(r_1 + r_2 + r_1r_2)^2$. ■

Theorem 3.4: Let G and H be disjoint graphs of order p_1 and p_2, and size q_1 and q_2, respectively. Let $G \times H$ denote the direct product of G and H. Then $w(G \times H) = w(G)w(H)$.

Proof: Let $(u, v) \in V(G \times H)$. Then

$$\begin{aligned} w(G \times H) &= \sum_{(u,v) \in V(G \times H)} (\deg_{G \times H}(u, v))^2 \\ &= \sum_{(u,v) \in V(G \times H)} (\deg_G(u)\deg_H(v))^2 \\ &= \sum_{u \in V(G)} (\deg_G(u))^2 \sum_{v \in V(H)} (\deg_H(v))^2 = w(G)w(H) \end{aligned}$$

■

Corollary 3.5: Let G and H be disjoint regular graphs of order p_1 and p_2, and degree r_1 and r_2, respectively. Then $w(G \times H) = p_1p_2(r_1r_2)^2$. ■

Theorem 3.6: Let G and H be disjoint graphs of order p_1 and p_2, and size q_1 and q_2, respectively. Let $G \circ H$ denote the lexicographic product of G and H. Then $w(G \circ H) = p_2^3 w(G) + p_1 w(H) + 8p_1q_1q_2$.

Proof: Let $(u, v) \in V(G \circ H)$. Then

$$\begin{aligned} w(G \circ H) &= \sum_{(u,v) \in V(G \circ H)} (\deg_{G \circ H}(u, v))^2 \\ &= \sum_{(u,v) \in V(G \circ H)} (|V(G)|\deg_G(u) + \deg_H(v))^2 \\ &= \sum_{(u,v) \in V(G \circ H)} [p_2^2(\deg_G(u))^2 + (\deg_H(v))^2 + 2p_2\deg_G(u)\deg_H(v)] \\ &= p_2^3 \sum_{u \in V(G)} (\deg_G(u))^2 + p_1 \sum_{v \in V(H)} (\deg_H(v))^2 + 2p_2 \sum_{u \in V(G)} \deg_G(u) \sum_{v \in V(H)} \deg_H(v) \\ &= p_2^3 w(G) + p_1 w(H) + 8p_1q_1q_2. \end{aligned}$$

■

Corollary 3.7: Let G and H be disjoint regular graphs of order p_1 and p_2, and degree r_1 and r_2, respectively. Then $w(G \circ H) = p_1p_2(p_1r_1 + r_2)^2$. ■

Open Problem: In addition to obtaining the weights of other basic graph classes, one can investigate the weights of graphs obtained by various types of composition that use more than two graphs at a time. See, for example, **[6]**. □

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[GTN LXII:3] STRONGLY *T*-SET GRACEFUL GRAPHS

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Abstract

The concept of a t-set graceful graph was introduced by B.D. Acharya. In this paper we show that a graph is t-set graceful if and only if it has a t-set graceful subgraph with the same t-number. Furthermore, we prove the existence of graphs for which every spanning subgraph is topologically set graceful, leading to the notion of strongly t-set graceful graphs. Several families of strongly t-set graceful graphs are obtained. Surprisingly, a graph is strongly t-set graceful only if the star graph of the same order is t-set graceful. We also characterize graphs for which all the spanning super graphs are t-set graceful.

1. Introduction

Set valuations of graphs, in which the vertices or edges are mapped to subsets of a given set subject to certain conditions, have been motivated by practical applications as well as intrinsic mathematical interest. B.D. Acharya pioneered the study of set valuations by introducing the notions of *set indexer number* and *topological set indexer number* for a given graph **[1]**. Subsequently, several authors studied set valuations of graphs and obtained many significant results; see for example, **[2]**–**[6]**. In **[7]**, the authors investigated topological set indexers and derived the topological numbers of certain graphs. An additional study **[8]** established the topological set gracefulness of certain stars, paths, and related graphs.

In this paper we examine the topological set gracefulness of subgraphs; in particular, spanning subgraphs of a *topologically set graceful* graph. It is proved that a graph G is t-set graceful if and only if G has a t-set graceful subgraph with the same t-number. We identify certain families of graphs for which every spanning subgraph is topologically set graceful. This leads to the notion of *strongly t-set graceful* graphs. On the other hand, if no spanning subgraph is t-set graceful, then G is called *weakly t-set graceful*. Several families of strongly t-set graceful graphs are obtained. Surprisingly, a graph is strongly t-set graceful only if the star graph of the same order is t-set graceful. We show that there are t-set graceful graphs that are neither strongly nor weakly t-set graceful. We characterize graphs for which all the spanning super graphs are t-set graceful.

2. Preliminaries

Throughout this paper, m and n denote natural numbers without restrictions (unless otherwise specified). For a nonempty set X, the set of all subsets of X is denoted by 2^X. The graphs we consider are all simple, with no loop or multiedge. We assume that readers are familiar with basic graph theory notation and definitions. To make the paper self-contained, less familiar definitions are included in this section.

Definition 1 [1]: Let $G = (V, E)$ be a graph and X be a nonempty set. A mapping $f\colon V \to 2^X$, or $f\colon E \to 2^X$, or $f\colon V \cup E \to 2^X$ is called a *set assignment* or *set valuation* of the vertices or edges or both. ■

Definition 2 [1]: Let G be a graph and let X be a nonempty set. A set valuation $f\colon V \cup E \to 2^X$ is said to be a *set indexer* of G if
1. $f(uv) = f(u)\Delta f(v)$, where Δ is the symmetric difference, and
2. the restrictive maps $f|_V$ and $f|_E$ are both injective. ■

In this case X is called an *index set* of G. Clearly a graph can have many index sets and the minimum of the cardinalities among all index sets is said to be the *set indexer number* of G and is denoted by $\gamma(G)$. The set indexer number of trivial graph K_1 is defined to be zero.

Definition 3 [1]: A set indexer f of a graph G with index set X is said to be a *topological set indexer* (or *t-set indexer*) if $f(V)$ is a topology on X. In this case X is called a *topological index set* (or *t-index set*) of G. The minimum number among the cardinalities of such topological index sets is said to be the *topological number* (*t-number*) of G and is denoted by $\tau(G)$. The corresponing t-set indexer is optimal. ■

Definition 4 [1]: A graph G is said to be *topologically set graceful* (or *t-set graceful*) if $\gamma(G) = \tau(G)$, where $\gamma(G)$ and $\tau(G)$ are, respectively, the set indexer number and t-number of G. ■

Definition 5 [1]: A graph G is said to be *set graceful* if $\gamma(G) = \log_2(|E| + 1)$. In this case the corresponding set indexer is called a *set graceful labeling* and it is the optimal set indexer of G. ■

3. Previous Results

In this section, for convenient access by the reader, we provide a list of previously published results that we use in this paper.

Theorem 3.1 [1]: Every graph has a set indexer. ■

Theorem 3.2 [1]: If X is an index set of a graph $G = (V, E)$, then
(1) $|E| \le 2^{|X|} - 1$, and
(2) $\lceil \log_2 |E| + 1 \rceil \le \gamma(G) \le |V| - 1$. ■

Theorem 3.3 [1]: If $G' \subseteq G$, then $\gamma'(G) \le \gamma(G)$. ■

Theorem 3.4 [1]: Every graph with at least two vertices has a t-set indexer. ■

Theorem 3.5 [1]: Let G be any graph with at least two vertices. Then $\gamma(G) \le \tau(G)$. ■

Theorem 3.6 [7]: If G' is a spanning subgraph of G, then $\tau(G') \le \tau(G)$. ■

Theorem 3.7 [7]: Let G be a graph of order m where $3 \cdot 2^{n-2} < m < 2^n$ for $n \ge 3$. Then $\tau(G) \ge n + 1$. ■

Theorem 3.8 [5]: $\gamma(K_{1,m}) = n + 1$ if and only if $2^n \le m < 2^{n+1}$. ■

Theorem 3.9 [5]: Let G be a graph of order n. Then $\gamma(K_{1,n-1}) \le \gamma(G)$. ■

Theorem 3.10 [8]: If $K_{1,n}$ is t-set graceful, then every spanning subgraph G of $K_{1,n}$ is t-set graceful. Moreover, $\tau(K_{1,n}) = \tau(G)$. ■

Theorem 3.11 [8]: $K_{1,n}$ is t-set graceful if it has a t-set graceful spanning subgraph G. Furthermore, $\tau(K_{1,n}) = \tau(G)$. ■

Theorem 3.12 [8]: The star graphs $K_{1,2^n+2^m-1}$, $0 \le m \le n$ are t-set graceful with t-number $n + 1$. ■

Theorem 3.13 [6]: $\gamma(P_m) = n + 1$, $2^n \le m \le 2^{n+1} - 1$. ■

Theorem 3.14 [1]: $\gamma(C_5) = 4$. ■

Theorem 3.15 [7]: $\tau(C_5) = 4$. ■

Theorem 3.16: $\gamma(C_6) = 4$. ■

Theorem 3.17 [1]: Let G be a graph with $\gamma(G) = |V| - 1$, then $|V| \leq 5$. ■

Theorem 3.18 [1]:

$$\gamma(K_n) = \begin{cases} n-1 \text{ if } 1 \leq n \leq 5 \\ n-2 \text{ if } 6 \leq n \leq 7. \end{cases}$$ ■

Theorem 3.19 [7]: $\tau(K_n) = n-1$, $n \geq 2$. ■

Theorem 3.20: $\tau(2K_2) = 3 = \gamma(2K_2)$. ■

Theorem 3.21 [9]: For integer $n \geq 2$, let $m(n)$ be the smallest positive integer such that there exists a topology on $m(n)$ points having n open sets.
Then $m(n) \leq \frac{4}{3}\lfloor \log_2 n \rfloor + 2$. ■

Theorem 3.22 [8][10]: The *double star graph, $ST(m, n)$*, formed from two stars $K_{1,m}$ and $K_{1,n}$ by joining their centers with an edge, where $0 \leq m \leq n-1$, is t-set graceful with t-number $n+1$. ■

Theorem 3.23 [11]: Every cycle graph C_{2^n-1}, $n \geq 2$ is set graceful. ■

Theorem 3.24 [8]: For the double star graph $ST(m, n)$, with $|V| = 2^l$, $l \geq 2$

$$\gamma(ST(m,n)) = \begin{cases} l \text{ if } m \text{ is even} \\ l+1 \text{ if } m \text{ is odd.} \end{cases}$$ ■

Theorem 3.25 [8]: The double star graph $ST(m, n)$,
with $3|V| = 2^k$, $k \geq 2$ is t-set graceful. ■

Theorem 3.26 [8]: $\gamma(P_n \cup K_1) = \gamma(P_n)$, $2^{m-1} \leq n < 2^m - 1$ and $m \geq 2$. ■

Theorem 3.27 [7]: $\tau(\overline{K_n}) = \tau(K_{1,n-1})$, $n \geq 2$. ■

Theorem 3.28 [8]: P_{2^n+2} is t-set graceful. ■

4. *T*-Set Graceful Subgraphs

In this section, we examine the topological set gracefulness of subgraphs, with special emphasis on spanning subgraphs. This suggests the concepts of *strongly* and *weakly t-set graceful graphs*. Conditions for strongly t-set graceful graphs are investigated and many families of such graphs are obtained.

Theorem 4.1 [7]: $\tau(K_n \cup K_1) = \tau(K_n)$, for all $n \geq 3$. ■

The order of a largest clique in a graph G is called the *clique number* of G and is denoted by $\mathrm{cl}(G)$ (see **[12]**).

Theorem 4.2: Let G be any graph with clique number $\mathrm{cl}(G) = n$ and order $|V| = n+1$, then $\tau(G) \geq n-1$. ■

Proof: Follows from Theorems 3.6, 3.19, and 4.1. ■

Remark: If $|V| > n+1$, then Theorem 4.2 may not hold. For example, consider $G = K_6 \cup 10K_1$ and $X = \{a, b, c, d\}$. Then by assigning $\varnothing, X, \{a\}, \{b\}, \{c\}, \{d\}$ to the vertices of K_6 and the remaining 10 subsets of X to the isolated vertices of G we obtain a t-set indexer of G that is optimal by Theorems 3.2 and 3.5.

Theorem 4.3: Let e_1, e_2, e_3 denote the edges of $K_{1,3} \subset K_n$, $n \geq 4$.
Then $\tau(K_n - \{e_1, e_2, e_3\}) = \tau(K_n) - 1$.

Proof: Let $G = K_n - \{e_1, e_2, e_3\}$, $V = \{u_1, u_2, \ldots, u_n\}$, and $e_1 = u_1u_2$, $e_2 = u_1u_3$, $e_3 = u_1u_4$. By Theorem 4.2, $\tau(G) \geq n-2$. Let $X = \{x_1, x_2, \ldots, x_{n-2}\}$. Now, by assigning $\{x_2\}$ to u_1 and $\varnothing, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \ldots, X$ to the vertices $u_2, u_3, \ldots, u_n$ of G, in that order, we obtain a t-set indexer of G. ■

Theorem 4.4: Let e_1, e_2, e_3 denote the edges of $K_{1,3} \subset K_n$, $n \geq 4$, and and let G be any graph such that $K_{n-1} \subseteq G \subseteq K_n - \{e_1, e_2, e_3\}$. Then $\tau(G) = n-2$.

Proof: Follows from Theorems 3.6, 3.19, 4.2, and 4.3. ■

Theorem 4.5: For $n \geq 9$, $\gamma(K_n) \leq n-3$.

Proof: Let $X = \{x_1, x_2, \ldots, x_{n-3}\}$, $n \geq 9$. By assigning

$$\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_{n-3}\}, \{x_1, x_2, x_3, x_4\}, \{x_3, x_4, x_5, x_6\}$$

to the vertices of K_n we obtain a set indexer of K_n. ■

Theorem 4.6: Let e_1, e_2, e_3 denote the edges of $K_{1,3} \subset K_n$, $n \geq 9$. Then $K_n - \{e_1, e_2, e_3\}$ is not t-set graceful.

Proof: Follows from Theorems 3.3, 4.3, and 4.5. ■

Theorem 4.7: A graph is t-set graceful if and only if it has a t-set graceful subgraph with the same t-number.

Proof: Since every graph is a subgraph of itself, the necessity part follows trivially.

Conversly, let $G' \subseteq G$ with $\tau(G') = \tau(G)$. Suppose G' is t-set graceful with t-number m. Then, by Theorems 3.3 and 3.5, $m = \gamma(G') \leq \gamma(G) \leq \tau(G) = \tau(G') = m$. Consequently, $\gamma(G) = \tau(G) = m$ and G is t-set graceful. ■

Remark: All subgraphs of a t-set graceful graph with the same t-number may not be t-set graceful. The join $K_1 \vee P_{n-1}$ of K_1 and the path graph P_{n-1} is called a *fan graph* and is denoted by F_n **[13]**. Consider the fan graph $F_7 = P_6 \vee \{u\}$, $P_6 = u_1 u_2 \ldots u_6$. By Theorems 3.2 and 3.5, $4 \leq \gamma(F_7) \leq \tau(F_7)$. Now define a t-set indexer f on F_7 with t-index set $X = \{x_1, x_2, x_3, x_4\}$ as follows: Assign

$$\varnothing, \{x_1\}, \{x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, X$$

to the vertices $u_1, u_2, \ldots, u_6, u$. Then, F_7 is t-set graceful with t-number 4. Since P_7 is a spanning subgraph of F_7, by Theorems 3.6 and 3.7, $4 \leq \tau(P_7) \leq \tau(F_7) = 4$. However, by Theorem 3.13, $\gamma(P_7) = 3$ so that P_7 is not t-set graceful. ■

Corollary 4.8: Let e_1, e_2, e_3 denote the edges of any $K_{1,3} \subset K_n$, $n \geq 9$ and let G be any graph such that $K_{n-1} \subseteq G \subseteq K_n - \{e_1, e_2, e_3\}$. Then G is not t-set graceful.

Proof: Follows from Theorems 4.4, 4.6, and 4.7. ■

Theorem 4.9: Let e_1 and e_2 be the edges of any $K_{1,2} \subset K_n$, $n \geq 3$, then $\tau(K_n - \{e_1, e_2\}) = n-1$.

Proof: The result is obvious when $n = 3$, thus we may assume that $n \geq 4$. Let $G = K_n - \{e_1, e_2\}$, $V = \{u_1, u_2, \ldots, u_n\}$, and $e_1 = u_1 u_2$, $e_2 = u_1 u_3$. By Theorem 4.2, $\tau(G) \geq n-2$. If possible, suppose that $\tau(G) = n-2$ and let f be an optimal t-set indexer with $X = \{x_1, x_2, \ldots, x_{n-2}\}$ as the t-index set. Now for any $u_i \neq u_j$, let $f(u_i) = A$ and $f(u_j) = B$. Because f is a t-set indexer, there exist vertices u_k and u_l such that $f(u_k) = A \cup B$ and $f(u_l) = A \cap B$.

<u>Claim</u>: $A \cup B = A$ or B.

If not, u_i, u_j, u_k, u_l are four distinct vertices of G. Clearly, $f(u_k u_l) = (A \cup B) \Delta (A \cap B) = A \Delta B = f(u_i u_j)$. Consequently, exactly one of $u_i u_j, u_k u_l \in E$. Without loss of generality, assume that $u_i u_j \notin E$ and $u_k u_l \in E$. Then, $f(u_i u_l) = f(u_j u_k) = A - B$, so that exactly one of $u_i u_l$ or $u_j u_k$, say $u_j u_k$, is in E. Also, $f(u_i u_k) = f(u_j u_l) = B - A$, so that exactly one of $u_i u_k$ or $u_j u_l$, say $u_j u_l$, is in E. Thus, all the three edges $u_i u_j$, $u_i u_k$, $u_i u_l$ do not belong to $G = K_n - \{e_1, e_2\}$, a contradiction. Hence, the claim.

Thus, for $i, j = 1, 2, 3, \ldots, n$, $f(u_i) \cup f(u_j) = f(u_i)$ or $f(u_j)$, so that we can order the vertices of G in such a way that $\varnothing = f(u_1) \subset f(u_2) \subset f(u_3) \subset \ldots \subset f(u_n) = X$. This shows that, $n-1 \geq |f(V)| = |V| = n$, a contradiction. Consequently, $\tau(G) \geq n-1$. However, G is a spanning subgraph of K_n. Thus, by Theorems 3.6 and 3.19, $\tau(G) = n-1$. ■

Theorem 4.10: Let e_1 and e_2 be the edges of any $K_{1,2} \subset K_n$, $n \geq 3$. Then, $\tau(K_n - \{e_1, e_2\}) = \tau(K_n - \{e_1\}) = \tau(K_n)$.

Proof: Follows from Theorems 3.19, 3.6, and 4.9. ■

Theorem 4.11 [1]: If G is a (p, q)-graph with $p \geq 6$, then $\gamma(G) \leq p - 2$. ■

Corollary 4.12: Let e_1 and e_2 be the edges of any $K_{1,2} \subset K_n$, $n \geq 6$. Then the graphs $K_n - \{e_1, e_2\}$ and $K_n - \{e_1\}$ are not *t*-set graceful.

Proof: Follows from Theorems 4.10 and 4.11. ■

Theorem 4.13: For any graph G, $\lceil \log_2(|V|) \rceil \leq \gamma(G)$.

Proof: Let G have order n. Then, by applying Theorems 3.2 and 3.9,

$$\gamma(G) \geq \gamma(K_{1,n-1}) \geq \lceil \log_2(|E(K_{1,n-1})| + 1) \rceil = \lceil \log_2(n - 1 + 1) \rceil = \lceil \log_2(n) \rceil = \lceil \log_2(|V|) \rceil .$$

■

Corollary 4.14: If $\tau(G) = \lceil \log_2(|V|) \rceil$, then G is *t*-set graceful.

Proof: Follows from Theorems 3.5 and 4.13. ■

Remark: The converse of Corollary 4.14 is not true in general. By Theorems 3.14 and 3.15 it follows that the cycle graph C_5 is *t*-set graceful. However, $\lceil \log_2(|V(C_5)|) \rceil = 3 < 4 = \tau(C_5)$. Thus, there are *t*-set graceful graphs G with $\tau(G) > \lceil \log_2(|V|) \rceil$.

Theorem 4.15 [1]: For any integer $n \geq 4$,

$$\gamma(K_n) \leq \left\lfloor \log_2\left(1 + \sum_{j=0}^{3} \binom{n-1}{j}\right) \right\rfloor .$$ ■

Theorem 4.16: For any *t*-set graceful graph G of order n,

$$\tau(G) \leq \left\lfloor \log_2\left(\frac{n^3 - 3n^2 + 8n + 6}{6}\right) \right\rfloor .$$

Proof: When $n = 2, 3$, by Theorems 3.5, 3.6, 3.19, and 4.13, $n - 1 \leq \gamma(G) \leq \tau(G) \leq \tau(K_n) = n - 1$. Assume that $n \geq 4$ and let G be a *t*-set graceful graph of order n. By Theorems 3.3 and 4.15,

$$\tau(G) = \gamma(G) \leq \gamma(K_n) \leq \left\lfloor \log_2\left(1 + \sum_{j=0}^{3} \binom{n-1}{j}\right) \right\rfloor .$$

Clearly,

$$\begin{aligned} 1 + \sum_{j=0}^{3} \binom{n-1}{j} &= 1 + \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} \\ &= 1 + 1 + n - 1 + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)(n-3)}{6} \\ &= \frac{n^3 - 3n^2 + 8n + 6}{6} \end{aligned}$$

■

Remark: The cycle graph C_6, the fan graph F_6, and the wheel graph W_5 (see **[14]**) are *t*-set graceful graphs attaining the above bound for $\tau(G)$.

In **[1]** B.D. Acharya obtained the set indexer number of K_n for $n \leq 7$. We have computed this number for K_n, $8 \leq n \leq 15$. The results are stated in the following theorem.

Theorem 4.17:

$$\gamma(K_n) = \begin{cases} 6 \text{ if } 8 \le n \le 9 \\ 7 \text{ if } 10 \le n \le 12 \\ 8 \text{ if } 13 \le n \le 15 \end{cases}$$

■

Theorem 4.18 [4]: $\gamma(K_n) \le \gamma(K_{n+1}) \le \gamma(K_n) + 1$, for all n. ■

Theorem 4.19: $\gamma(K_n) \ge \frac{4}{3}\lfloor \log_2 n \rfloor + 2$, $n \ge 7$. ■

Proof: By Theorem 3.18, $\gamma(K_7) = 5$. For $8 \le n \le 15$, by Theorem 4.17, $\gamma(K_n) \ge 6$. For $16 \le n \le 31$, by Theorems 4.17 and 4.18, $\gamma(K_n) \ge 8$. Now, for $m \ge 5$, choose an integer n such that $2^m \le n \le 2^{m+1} - 1$. By Theorem 3.2,

$$\begin{aligned} \gamma(K_n) &\ge \left\lceil \log_2\left(\frac{n(n-1)}{2} + 1\right) \right\rceil \\ &\ge \left\lceil \log_2\left(\frac{n(n-1)}{2}\right) \right\rceil \\ &= \lceil \log_2(n) + \log_2(n-1) - 1 \rceil \\ &\ge 2m - 1 > m + \left(\frac{m}{3} + 2\right) \\ &= \frac{4m}{3} + 2 = \frac{4}{3}\lfloor \log_2 n \rfloor + 2 \end{aligned}$$

■

By combining Theorems 4.15 and 4.19 we obtain the following result.

Corollary 4.20:

$$\frac{4}{3}\lfloor \log_2 n \rfloor + 2 \le \gamma(K_n) \le \left\lfloor \log_2\left(\frac{n^3 - 3n^2 + 8n + 6}{6}\right) \right\rfloor, \ n \ge 7.$$ ■

Theorem 4.21: $n \le \tau(K_{1,m}) \le \frac{4n}{3} + 2$, $2^n - 1 \le m \le 2^{n+1} - 2$.

Proof: Let X be a set of minimum cardinality so that there is a topology τ on X with $m + 1$ open sets. Then by Theorem 3.21, $n \le |X| \le 4n/3 + 2$. Now by assigning the open sets of τ to the vertices of $\overline{K}_{m+1}$, we obtain a t-set indexer of $\overline{K}_{m+1}$. However, by Theorem 3.27, $\tau(\overline{K}_{m+1}) = \tau(K_{1,m})$. Consequently, $n \le \tau(K_{1,m}) \le \frac{4n}{3} + 2$. ■

Corollary 4.22: $\tau(K_{1,m}) \le \gamma(K_{m+1})$.

Proof: For $m \ge 1$, there exists an n such that $2^n \le m + 1 \le 2^{n+1} - 1$. For $1 \le m \le 5$, by Theorems 3.12 and 3.18, $\tau(K_{1,m}) \le \gamma(K_{m+1})$. If $m \ge 6$, from Theorems 4.19 and 4.21, $\tau(K_{1,m}) \le n + 2 + n/3 \le \gamma(K_{m+1})$. ■

Conjecture 1: For any $n > 1$, there exists a t-set graceful nonempty graph of order n. □

Theorem 4.23: The wheel graph W_6 is t-set graceful with t-number 4.

Proof: Let $W_6 = C_6 \vee \{u\}$, $C_6 = u_1 u_2 \ldots u_6 u_1$. By Theorems 3.2 and 3.5, $\tau(W_6) \ge 4$. However, we can define a t-set indexer f on W_6 with t-index set $X = \{a, b, c, d\}$ as follows: Assign $\emptyset, \{a\}, \{a, b, d\}, \{b\}, \{a, b, c\}, \{a, b\}, X$ to the vertices $u, u_1, u_2, \ldots, u_6$, in that order. Clearly, $f(V)$ is a topology on X. ■

Theorem 4.24: Let G be a graph of order 7 such that $C_7 \subset G \subseteq W_6$.
Then G is *t*-set graceful.

Proof: By Theorems 3.2, 3.5, 3.6, and 4.23, $4 \le \gamma(G) \le \tau(G) \le \tau(W_6) = 4$.
Thus, G is *t*-set graceful. ■

Theorem 4.25: A graph G with *t*-number $|V| - 1$ is *t*-set graceful if only if $2 \le |V| \le 5$.

Proof: Let G be a *t*-set graceful graph with $\tau(G) = |V| - 1$. Then, by Theorems 3.4 and 3.17, $2 \le |V| \le 5$.

Conversely, suppose $\tau(G) = |V| - 1$, $2 \le |V| \le 5$. Then, the following four cases are possible.

Case-1: $|V| = 2$. Then $G = K_2$ or $\bar{K}_2$. Now by applying Theorems 3.5, 3.6, 3.19, and 4.13,

$$1 = \lceil \log_2 2 \rceil \le \gamma(G) \le \tau(G) \le \tau(K_2) = 1.$$

Case-2: $|V| = 3$. Then G is a spanning subgraph of K_3 and by applying Theorems 3.5, 3.6, 3.19, and 4.13,

$$2 = \lceil \log_2 3 \rceil \le \gamma(G) \le \tau(G) \le \tau(K_3) = 2.$$

Case-3: $|V| = 4$. Then G is a spanning subgraph of K_4 and also a spanning super graph of $2K_2$. Now, by applying Theorems 3.3, 3.5, 3.6, 3.19, and 3.20

$$3 = \gamma(2K_2) \le \gamma(G) \le \tau(G) \le \tau(K_4) = 3.$$

Case-4: $|V| = 5$. Then G is a spanning subgraph of K_5 and also a spanning super graph of C_5. Now, by applying Theorems 3.3, 3.5, 3.6, 3.14, and 3.19

$$4 = \gamma(C_5) \le \gamma(G) \le \tau(G) \le \tau(K_5) = 4.$$ ■

Corollary 4.26: K_n is *t*-set graceful if and only if $2 \le n \le 5$.

Proof: Follows from Theorems 3.19 and 4.25. ■

Remark: There are *t*-set graceful graphs G for which $\tau(G) \ne |V| - 1$, $2 \le |V| \le 5$. For example, P_5 is *t*-set graceful with *t*-number 3.

Theorem 4.27: Every spanning supergraph of a *t*-set graceful graph is *t*-set graceful if and only if $2 \le |V| \le 5$.

Proof: The necessary part follows from Corollary 4.26 since every graph with $n \ge 6$ vertices has K_n, as a spanning super graph.

Conversely, let G be a *t*-set graceful graph with $2 \le |V| \le 5$. We prove that any spanning supergraph G' of G is *t*-set graceful. Suppose $|V| = 2$ or 3. Then, by Theorems 4.13, 3.5, 3.6, and 3.19,

$$|V| - 1 = \lceil \log_2 |V| \rceil \le \gamma(G') \le \tau(G') \le \tau(K_n) = |V| - 1.$$

Suppose $|V| = 4$ or 5. Then by Theorems 4.13 and 3.2, $|V| - 2 = \lceil \log_2 |V| \rceil \le \gamma(G') \le |V| - 1$ so that $\gamma(G') = |V| - 1$ or $|V| - 2$.

Case-1: $\gamma(G') = |V| - 1$. Then, by Theorems 3.5, 3.6, and 3.19, $|V| - 1 \le \gamma(G') \le \tau(G') \le \tau(K_n) = |V| - 1$.

Case-2: $\gamma(G') = |V| - 2$.

Subcase 2.1: $|V| = 4$. Then $\gamma(G') = 2$ and by Theorem 3.2, $|E(G')| \le 3$. Also, by Theorems 3.3 and 3.13, P_4 is not subgraph of G'. Consequently, G' is a spanning subgraph of $H = K_4 - \{e_1, e_2, e_3\}$, where $\langle e_1, e_2, e_3 \rangle = K_{1,3}$ or K_3. Then, by Theorems 3.5 and 3.6, $2 = \gamma(G') \le \tau(G') \le \tau(H) = 2$.

Subcase 2.2: $|V| = 5$. Consider the complete graph K_5. Let $H_1 = K_5 - \{e_1, e_2, e_3, e_4\}$, where $\langle e_1, e_2, e_3 \rangle = K_{1,3}$; $H_2 = K_5 - \{e_1, e_2, e_3\}$, where $\langle e_1, e_2, e_3 \rangle = K_3$; and let H_3 be any proper spanning subgraph of $K_5 - \{e_1, e_2, e_3, e_4\}$, where $\langle e_1, e_2, e_3, e_4 \rangle = C_4$. It can be easily verified that $\tau(H_1) = \tau(H_2) = \tau(H_3) = 3$. Since $\gamma(G') = 3$, by Theorem 3.2, $|E(G')| \le 7$. By Theorems 3.3 and 3.14, $C_5 \not\subset G'$. Consequently, G' is a spanning subgraph of H_1 or H_2 or H_3. Then, by Theorems 3.5 and 3.6, $3 = \gamma(G') \le \tau(G') \le \tau(H_i) = 3$, $i = 1, 2, 3$. ■

Corollary 4.28: Every graph G with $2 \le |V| \le 5$ is t-set graceful.

Proof: By Theorems 3.10 and 3.12 it follows that $\bar{K}_n$ is t-set graceful for $2 \le n \le 5$. Thus, the corollary follows from Theorem 4.27. ■

Remark: Every proper spanning subgraph of a t-set graceful graph need not be t-set graceful. Consider the graph $G = C_6 \cup \{u_7\}$, $C_6 = u_1 u_2 \dots u_6 u_1$. By Theorems 3.3, 3.5, and 3.16, $4 \le \gamma(G) \le \tau(G)$. We can define a t-set indexer f on G with t-index set $X = \{x_1, x_2, x_3, x_4\}$ as follows: Assign $X, \varnothing, \{x_1\}, \{x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}$ to the vertices $u_1, u_2, \dots, u_7$, in that order. Let G' be any proper spanning subgraph of G. Then, by Theorems 3.3, 3.6, 3.7, 3.13, and 4.13, $3 = \gamma(G') < \tau(G') \le \tau(G) = 4$, so that G' is not t-set graceful.

Unlike the case with spanning supergraphs, there are a variety of graphs for which every spanning subgraph is t-set graceful and this motivates the following.

Definition 6: A graph G is said to be *strongly t-set graceful*, if every spanning subgraph of G is t-set graceful. ■

Obviously strongly t-set graceful graphs are t-set graceful. Furthermore, G is strongly t-set graceful if and only if every spanning subgraph of G is strongly t-set graceful.

Theorem 4.29: $K_{1,n}$ is t-set graceful if and only if it is strongly t-set graceful.

Proof: Follows from Theorem 3.10. ■

Theorem 4.30: If $K_{1,n}$ is not t-set graceful, then no graph of order $n+1$ is strongly t-set graceful.

Proof: Since $K_{1,n}$ is not t-set graceful, by Theorem 3.11, no spanning subgraph of $K_{1,n}$ is t-set graceful. In particular, $\bar{K}_{n+1}$ is not t-set graceful. ■

Remark: The previous theorem states that the t-set gracefulness of the star graph of the same order is a necessary condition for a given graph to be strongly t-set graceful. However, the converse is not true. By Corollary 4.26, K_8 is not strongly t-set graceful. However, $K_{1,7}$ is t-set graceful by Theorem 3.12.

Corollary 4.31: No graph of order m, $3 \cdot 2^{n-2} < m < 2^n$ $(n > 2)$ is strongly t-set graceful.

Proof: By Theorems 3.7 and 3.8, $K_{1,m-1}$ is not t-set graceful. Thus, the result follows from Theorem 4.30. ■

Theorem 4.32: Every graph of order m, $2 \le m \le 5$ is strongly t-set graceful.

Proof: Follows from Corollary 4.28. ■

Theorem 4.33: Let e_1, e_2, e_3 be the edges of any $K_{1,3} \subset K_5$. Then the graph $H = K_5 - \{e_1, e_2, e_3\} \cup K_1$ is strongly t-set graceful.

Proof: Let $u_1, \dots, u_6$ be vertices of H such that $d(u_1) = d(u_2) = d(u_3) = 3$, $d(u_4) = 4$, $d(u_5) = 1$, and $d(u_6) = 0$. By Theorems 4.13 and 3.5, $3 \le \gamma(H) \le \tau(H)$. Let $X = \{a, b, c\}$. By assigning $\{a\}, \{a, b\}, X, \varnothing, \{a, c\}, \{b\}$ to the vertices $u_1, \dots, u_6$, in that order, we obtain a t-set indexer of H with X as the t-index set. The result follows from Theorems 4.13, 3.5, and 3.6. ■

Theorem 4.34: Every graph G with $\tau(G) = \lceil \log_2 |V| \rceil$ is strongly t-set graceful.

Proof: Let G' be a spanning subgraph of G. Then, by applying Theorems 4.13, 3.3, and 3.5,

$$\lceil \log_2 |V| \rceil \le \gamma(G') \le \gamma(G) \le \tau(G) = \lceil \log_2 |V| \rceil.$$

Furthermore, from Theorems 4.13, 3.5, and 3.6,

$$\lceil \log_2 |V| \rceil \le \gamma(G') \le \tau(G') \le \tau(G) = \lceil \log_2 |V| \rceil.$$

Thus, $\gamma(G') = \tau(G') = \lceil \log_2 |V| \rceil$, so that G' is t-set graceful. ■

Theorem 4.35: Every t-set graceful path P_n, $n \ne 2^m$ is strongly t-set graceful.

Proof: Follows from Theorems 3.13 and 4.34. ■

Theorem 4.36: Every tree of order 6 is strongly t-set graceful.

Proof: By Theorems 3.12, 3.10, 3.28, and 4.35, $K_{1,5}$, P_6, and the two double stars of order 6 are strongly t-set graceful. Let $u_1, \ldots, u_6$ be the vertices of the two remaining trees (the caterpillars of diameter 4), call them T_1 and T_2. Let u_1, u_2, u_3, u_4, u_5 be the longest path in both T_1 and T_2, and let $u_2u_6 \in T_1$ and $u_3u_6 \in T_2$. Let $X = \{a, b, c\}$. Then, by assigning $\{a, c\}, \emptyset, \{a, b\}, X, \{a\}, \{b\}$ to the vertices $u_1, \ldots, u_6$ of T_1, in that order, and by assigning $\{a, c\}, \{b\}, \emptyset, \{a, b\}, X, \{a\}$ to the vertices $u_1, \ldots, u_6$ of T_2, in that order, we obtain t-set indexers of T_1 and T_2 with X as the t-index set. Now by Theorems 4.13 and 3.5, $3 = \lceil \log_2|V| \rceil \le \gamma(T_i) \le \tau(T_i) \le 3$ for $i = 1, 2$. Consequently, both T_1 and T_2 are strongly t-set graceful by Theorem 4.34. ■

Corollary 4.37: Every tree of order at most 6 is strongly t-set graceful.

Proof: Follows from Theorem 4.36 and Corollary 4.28. ■

Example 1: C_6 is strongly t-set graceful. Let $C_6 = v_1 \ldots v_6 v_1$. By Theorems 3.5 and 3.16, $\tau(C_6) \ge \gamma(C_6) = 4$. Let $X = \{a, b, c, d\}$. By assigning $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X, \{a, c\}$ to the vertices $v_1, \ldots, v_6$, in that order, we obtain a t-set indexer of C_6 with t-index set X. Thus, C_6 is t-set graceful and the result follows from Theorem 4.36.

Example 2: $C_5 \cup K_1$ is strongly t-set graceful. Let $C_5 = u_1 u_2 \ldots u_5 u_1$ and $K_1 = \{u\}$. By Theorems 3.14, 3.3, and 3.5, $4 \le \gamma(C_5) \le \gamma(C_5 \cup K_1) \le \tau(C_5 \cup K_1)$. Now define a t-set indexer f on $C_5 \cup K_1$ with t-index set $X = \{a, b, c, d\}$ as follows: Assign $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X, \{b\}$ to the vertices $u_1, \ldots, u_5$ and u, in that order. Thus, $C_5 \cup K_1$ is t-set graceful and by Theorem 4.36, $C_5 \cup K_1$ is strongly t-set graceful.

Example 3: $K_4 \cup K_2$ is strongly t-set graceful. Let $V = \{u_1, u_2, \ldots, u_6\}$ where $K_2 = u_5u_6$. By Theorem 4.13, $\gamma(K_4 \cup K_2) \ge 3$. Let $X = \{a, b, c\}$. Then, $\{a\} = \emptyset \Delta \{a\} = \{a, b\} \Delta \{b\} = \{a, c\} \Delta \{c\} = X \Delta \{b, c\}$ and $\{b\} = \emptyset \Delta \{b\} = \{a, b\} \Delta \{a\} = \{b, c\} \Delta \{c\} = X \Delta \{a, c\}$. Consequently, there does not exist a set indexer of $K_4 \cup K_2$ with X as the index set. However, by assigning $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$, $Y = \{a, b, c, d\}$, and $\{b\}$ to the vertices $u_1, \ldots, u_6$, in that order, we obtain a t-set indexer f of $K_4 \cup K_2$ with Y as the t-index set. Thus, $K_4 \cup K_2$ is t-set graceful with t-number 4.

Now consider the spanning subgraphs G_1 and G_2 of $K_4 \cup K_2$ given by $G_1 = K_4 \cup K_2 - \{u_1u_3\}$ and $G_2 = K_4 \cup K_2 - \{u_5u_6\}$. Clearly, every spanning subgraph of $K_4 \cup K_2$ is either a spanning subgraph of G_1 or G_2. By assigning $\emptyset, \{a, b\}, X, \{a\}, \{b\}, \{a, c\}$ to $u_1, \ldots, u_6$, in that order, we obtain t-set indexers for G_1 and G_2 with X as the t-index set. Thus, by Theorems 4.13 and 3.5, G_1 and G_2 are t-set graceful with t-number $\lceil \log_2|V| \rceil = 3$. The result now follows from Theorem 4.34. ■

Example 4: $2K_3$ is strongly t-set graceful. Let $G = 2K_3 = u_1u_2u_3u_4 \cup v_1v_2v_3v_4$. By Theorem 4.13, $\gamma(G) \ge 3$. Consider the set $X = \{x_1, x_2, x_3\}$. Since

$$\{A_1 \Delta A_2, A_1 \Delta A_3, A_2 \Delta A_3\} \cap \{B_1 \Delta B_2, B_1 \Delta B_3, B_2 \Delta B_3\} \neq \emptyset, \ \forall A_i, B_i \in 2^X, \ i = 1, 2, 3,$$

there is no set indexer of G with X as the index set. Hence, $\gamma(G) \ge 4$. By assigning $\emptyset, \{a\}, \{a, b\}, \{b\}, \{a, b, c\}$, and $Y = \{a, b, c, d\}$ to the vertices u_1, u_2, u_3, v_1, v_2, and v_3, in that order, we obtain a t-set indexer of G with Y as the t-index set. Thus, G is t-set graceful with t-number 4.

Let $H = G - \{u_1u_2\}$. Assign $\emptyset, \{b\}, \{a, c\}, \{a\}, \{a, b\}, X$ to the vertices $u_1, u_2, u_3, v_1, v_2, v_3$, in that order, we obtain a t-set indexer of H. Then, by Theorems 4.13 and 3.5, $3 = \lceil \log_2|V| \rceil \le \gamma(H) \le \tau(H) = 3$ and hence, H is t-set graceful with t-number $\lceil \log_2|V| \rceil = 3$. Consequently, G is strongly t-set graceful by Theorem 4.34.

Remark: For the strongly t-set graceful graphs discussed in the previous examples, $\tau(G) > \lceil \log_2|V| \rceil$. Thus, the converse of Theorem 4.34 is not true.

Corollary 4.38: If $P_n \cup K_1$, $n \ge 2$ is t-set graceful, then it is strongly t-set graceful.

Proof: Follows from Theorems 4.34 and 3.26. ■

Theorem 4.39: The double star of order $2^n + 2^m$, $m, n \ge 0$ is strongly t-set graceful.

Proof: Let G be a double star of order $2^n + 2^m$.

<u>Case-1</u>: $m = n$. Then $|V| = 2^{n+1}$.

Subcase 1.1: Both m and n are even. Then, by Theorem 3.25, G is t-set graceful and by Theorem 3.24,

$$\gamma(G) = \tau(G) = n+1 = \lceil \log_2|V| \rceil .$$

Now, by Theorem 4.34, every spanning subgraph of G is t-set graceful.

Subcase 1.2: Both m and n are odd. By Theorem 3.25, G is t-set graceful and by Theorem 3.24, $\gamma(G) = \tau(G) = n+2$. Let G' be any spanning subgraph of G. Then, by Theorems 4.13 and 3.3,

$$\lceil \log_2|V| \rceil = \lceil \log_2 2^{n+1} \rceil = n+1 \le \gamma(G') \le \gamma(G) = n+2 .$$

Thus, $\gamma(G') = n+1$ or $n+2$.

If $\gamma(G') = n+1$, then the optimal set indexer f of G' is a t-set indexer forming the discrete topology $f(V)$ on the corresponding index set.

If $\gamma(G') = n+2$, then by Theorems 3.5 and 3.6, it follows that $n+2 = \gamma(G') \le \tau(G') \le \tau(G) = n+2$.

Thus, if $m = n$, then every spanning subgraph of G is t-set graceful with t-number either $n+1$ or $n+2$.

<u>Case-2</u>: $m \ne n$. By Theorem 3.22, $\gamma(G) = \tau(G) = n+1$. Let G' be any spanning subgraph of G. Then by Theorems 4.13, 3.5, and 3.6, $n+1 = \lceil \log_2|V| \rceil \le \gamma(G') \le \tau(G') \le \tau(G) = n+1$. Thus, in this case every spanning subgraph of G is t-set graceful with t-number $n+1$. ■

Theorem 4.40 [6]: $\gamma(C_{2^n-1} \cup K_1) = n$, $n \ge 2$. ■

Theorem 4.41: $C_{2^n-1} \cup K_1$ is strongly t-set graceful.

Proof: Let G' be any spanning subgraph of $G = C_{2^n-1} \cup K_1$. Then, by Theorems 4.13, 3.3, and 4.40, $n = \lceil \log_2|V| \rceil \le \gamma(G') \le \gamma(G) = n$. Thus, $\gamma(G') = n$ and the optimal set indexer f defines a discrete topology $f(V)$ on the corresponding index set. Consequently, G' is t-set graceful. ■

Remark: Set graceful and t-set graceful are independent notions. For example, C_{2^n-1} is set graceful but not t-set graceful; whereas P_5 is t-set graceful but not set graceful. Furthermore, $C_{2^n-1} \cup K_1$ is both set graceful and t-set graceful.

The following theorem provides a class of graphs for which neither the graph nor its spanning subgraphs are t-set graceful.

Theorem 4.42: Spanning subgraphs of C_{2^n-1}, $n \ge 3$ are not t-set graceful.

Proof: Follows from Theorems 3.3, 3.7, and 3.23. ■

Conjecture 2: C_{2^n} is strongly t-set graceful. □

Remark: There are t-set graceful graphs for which no proper spanning subgraph is t-set graceful. Consider the graph $G = C_6 \cup K_1$. By Theorems 3.16, 3.3, 3.5, 3.6, and the Remark following Theorem 4.7

$$4 = \gamma(C_6) \le \gamma(G) \le \tau(G) \le \tau(F_7) = 4$$

and hence, G is t-set graceful with t-number 4. Consider $P_6 \cup K_1$, then by Theorems 4.13, 3.3, 3.5, and 3.6,

$$3 \le \gamma(P_6 \cup K_1) \le \gamma(P_7) = 3 < \tau(P_6 \cup K_1) .$$

Clearly, $P_6 \cup K_1$ is a not t-set graceful. Let G' be any proper spanning subgraph of G. Clearly, $G' \subseteq G$ and by Theorems 4.13, 3.3, and 3.7, $3 \le \gamma(G') \le \gamma(P_6 \cup K_1) = 3 < \tau(G')$. Thus, G' is not t-set graceful.

This motivates the following definition.

Definition 7: A t-set graceful graph is said to be *weakly t-set graceful* if no proper spanning subgraph of it is t-set graceful. ■

Any nonempty strongly t-set graceful graph is not weakly t-set graceful. There are empty graphs $\bar{K}_{10}, \bar{K}_9, \bar{K}_8, \bar{K}_6, \bar{K}_5$ that are both strongly and weakly t-set graceful.

Example 5: $C_5 \cup 2K_1$ is weakly t-set graceful. By Theorems 3.14, 3.3, 3.5, 3.6, and the Remark following Theorem 4.7,

$$4 \le \gamma(C_5) \le \gamma(C_5 \cup 2K_1) \le \tau(C_5 \cup 2K_1) \le \tau(F_7) = 4 .$$

Let G' be any proper spanning subgraph of $C_5 \cup 2K_1$. Clearly, $G' \subseteq P_7$ and by Theorems 4.13, 3.3, 3.13, and 3.7, $3 \le \gamma(G') \le \gamma(P_7) = 3 < \tau(G')$. Thus, G' is not *t*-set graceful. Hence, $C_5 \cup 2K_1$ is weakly *t*-set graceful.

Remark: There are *t*-set graceful graphs that are neither strongly nor weakly *t*-set graceful. By the Remark following Theorem 4.7, F_7 is a *t*-set graceful graph that is not strongly *t*-set graceful. However, by Theorem 4.24 it follows that F_7 is not weakly *t*-set graceful.

Theorem 4.43: Every graph *G* of order $2^n + 2^m$, $m, n \ge 0$ have at least one *t*-set graceful spanning subgraph.

Proof: By Theorem 3.12, $K_{1,\, 2^n + 2^m - 1}$ is *t*-set graceful and $\bar{K}_{2^n + 2^m} \subset K_{1,\, 2^n + 2^m - 1}$. However, by Theorem 3.10, $\bar{K}_{2^n + 2^m}$ is *t*-set graceful. ■

Remark: Every graph with at least two vertices contains the *t*-set graceful subgraph $\bar{K}_2$. However, not all graphs with at least two vertices contain a *t*-set graceful spanning subgraph. Consider the path P_7. By Theorem 3.13, $\gamma(P_7) = 3$. Let *G* be any spanning subgraph of P_7. Then, by Theorems 3.3, 3.7, 3.13, and 4.13, $3 = \lceil \log_2 7 \rceil \le \gamma(G) \le \gamma(P_7) = 3 < \tau(G)$. Consequently, *G* is not *t*-set graceful.

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[GTN LXII:4] SELF VERTEX SWITCHINGS OF UNICYCLIC GRAPHS

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Abstract

A vertex $v \in V(G)$ in a graph G is said to induce a self vertex switching in G if G is isomorphic with G^v, where G^v is the graph obtained from G by deleting all edges of G incident with v and adding edges to connect v with each vertex that is not adjacent to v in G. Elsewhere, the author characterizes trees with a self vertex switching. In this paper we characterize unicyclic graphs with a self vertex switching.

1. Introduction

For a finite undirected graph $G(V, E)$ with order $|V(G)| = p$ and a set $\sigma \subseteq V$, the *switching* of G by σ is defined to be the graph $G^{\sigma}(V, E')$, obtained from G by removing all edges between σ and its complement $V - \sigma$ and inserting an edge uv for each pair $u \in \sigma$ and $v \in V - \sigma$ that are not adjacent in G. Switching was defined by Seidel **[1]** and is also referred to as *Seidel switching*. If $\sigma = \{v\} \subset V$, we call the corresponding switching $G^{\{v\}}$ a *vertex switching* and denote it by G^v **[2]**. A subset σ of $V(G)$ is called a *self switching* of G if $G \cong G^{\sigma}$. The set of all self switchings of G with cardinality k is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. If $k = 1$, we use the term *self vertex switching* **[1][3]**.

A *branch* at a vertex v in G is a connected subgraph B of G such that $B - v$ is connected and maximal **[3]**. A branch B at v in G is said to be a *self switching branch* if $B \cong B^v$. In G, two branches B_1 and B_2 at v are said to be *complementary switching branches* if there exist isomorphisms f_1 between B_1 and B_2^v and f_2 between B_2 and B_1^v such that $f_1(v) = f_2(v)$ **[3]**.

Elsewhere, **[4]**, we characterize trees and forests with a self vertex switching. Here, we characterize unicyclic graphs with a self vertex switching. The following results are required in the subsequent sections.

Theorem 1.1 [2]: If v is a self vertex switching of a graph G of order p, then p is odd and $d_G(v) = (p-1)/2$. ■

Theorem 1.2 [4]: Let v be a self vertex switching of a connected graph G and let B be a branch at v in G. Then, $|V(B)| \geq 3$. ■

Theorem 1.3. [3]: Let $k(G)$ be the number of components of G and let B_i be either a branch at v in G or the union of v and a component of G not containing v, for $i = 1, 2, \ldots, k(G-v)$. Then, $G = \bigcup_{i=1}^{k} B_i$ and $G^v = \bigcup_{i=1}^{k} B_i^v$. ■

Corollary 1.4 [3]: Let v be any vertex of a connected graph G such that G^v is connected. Then, B is a branch at v in G if and only if B^v is a branch at v in G^v. ■

Lemma 1.5 [4]: A subgraph D is a component of a graph G not containing v if and only if the join $D + v$ is a branch at v in G^v. ■

Theorem 1.6 [4]: Let v be any vertex of a nontrivial graph G. Then, G^v is connected if and only if $d_G(v) = 0$ or $d_B(v) < |V(B)| - 1$ for every branch B at v in G. ■

Theorem 1.7 [4]: Let v be any vertex of a nontrivial connected graph G. Then, G^v is a tree if and only if $G - v$ is acyclic and $d_B(v) = |V(B)| - 2$ for every branch B at v in G. ■

Theorem 1.8 [4]: Let v be a vertex of a nontrivial graph G. Then, G^v is a disconnected graph with k components if and only if G has at least $k-1$ branches at v and $d_B(v) = |V(B)| - 1$ for only $k-1$ branches B at v in G. ■

2. Self Vertex Switchings of Connected Unicyclic Graphs

Let v be a vertex of a graph G and G^v the switching of G by v. Theorem 1.6 gives conditions on the vertex v of G such that G^v is connected. In this Section we characterize vertices v of G such that G^v is connected and unicyclic. Using this, we characterize connected unicyclic graphs with a self vertex switching.

Theorem 2.1: Let v be a non cutvertex of a graph G of order $p \geq 3$. Let $D \neq K_1, K_2$ be a component of G that contains v. Then, G^v is connected and unicyclic if and only if one of the following statements is true:
(1) $G = K_2 \cup (p-2)K_1$ and v is one of the K_1s.
(2) G is connected, $G - v$ is acyclic, and $d_G(v) = |V(G)| - 3$.
(3) G is connected, $G - v$ is unicyclic, and $d_G(v) = |V(G)| - 2$.
(4) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, and $d_G(v) = |V(D)| - 2$.
(5) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is acyclic, and $d_G(v) = |V(D)| - 3$.
(6) $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, $G - v$ is acyclic, and $d_G(v) = |V(D)| - 2$.

Proof: Let G^v be connected and unicyclic. Then $G - v$ is either acyclic or unicyclic. Using Theorem 1.6, $d_G(v) = 0$ or $d_B(v) < |V(B)| - 1$ for every branch B at v in G since G^v is connected.

If $d_G(v) = 0$, then G is disconnected. Since G^v is unicyclic, we obtain $G = K_2 \cup (p-2)K_1$, where v is one of the K_1s. Hence, (1) is proved.

Now consider the case $d_G(v) \neq 0$ and $d_B(v) < |V(B)| - 1$. That is, when $d_G(v) \neq 0$ and $d_B(v) \leq |V(B)| - k$, $k \geq 2$. Since v is not a cutvertex of G, B is either G or a component of G containing v (depending on whether G is connected or disconnected). There are four cases.

Case 1: G is connected and $G - v$ is acyclic.

In this case, $B = G$. Since v is not a cutvertex of G, $G - v$ is connected. If $d_G(v) = |V(G)| - 3$, then there are precisely two vertices, u and w, that are not adjacent to v in G. Since $G - v$ is connected and acyclic, there exists a unique $u—w$ path in $G - v$, and hence, also in G^v. Therefore, the edge vu, the path $u—w$, and the edge wv form a unique cycle in G^v. If $d_G(v) = |V(G)| - 2$, then G^v is a tree, by Theorem 1.7. If $d_G(v) < |V(G)| - 3$, then there are at least three vertices that are not adjacent to v in G, and hence, G has at least two edges that are not incident on v. Consequently, G^v has at least two cycles, which is a contradiction. Therefore, $d_G(v) = |V(G)| - 3$, proving Statement (2).

Case 2: G is connected and $G - v$ is unicyclic.

In this case, once again $B = G$. Let C be the unique cycle in $G - v$. Then, C is also a cycle of G^v that does not contain v. By Theorem 1.6, $d_G(v) \leq |V(G)| - 2$ since G^v is connected. Suppose $d_G(v) < |V(G)| - 2$. Then there are at least two vertices, u and w, that are not adjacent to v in G. Since $G - v$ is connected, there exists a $u—w$ path in $G - v$, and hence, also in G^v. The edges wv and vu and the path $u—w$ form a cycle in G^v, different from C. This contradicts our assumption that G^v is unicyclic. Consequently, $d_G(v) = |V(G)| - 2$ and Statement (3) follows.

Case 3: G is disconnected and $G - v$ is unicyclic.

Clearly, v is in a nontrivial component, D, of G since otherwise G^v is not unicyclic. The other components of G are trivial graphs. This implies that $G = D \cup (p - |V(D)|)K_1$, so that, $G^v = D^v \cup [v + (p - |V(D)|)K_1]$. Since G^v is connected and unicyclic, D^v is also connected and unicyclic. Furthermore, since $G - v$ is unicyclic, then so is $D - v$. Applying Case 2 to D, we obtain $d_D(v) = |V(D)| - 2$. Thus, $G = D \cup (p - |V(D)|)K_1$ and $d_G(v) = |V(G)| - 2$, proving Statement (4).

Case 4: G is disconnected and $G - v$ is acyclic.

In this case, v is in a nontrivial component, D, of G. Since G^v is connected and D is the only branch at v in G, $d_G(v) = d_D(v) \leq |V(D)| - 2$. If $d_D(v) < |V(D)| - 3$, then, as in Case 1, we can prove that D^v is not unicyclic, and hence, G^v is not unicyclic. This implies that $d_G(v) = |V(D)| - 2$ or $d_G(v) = |V(D)| - 3$.

When $d_G(v) = |V(D)| - 3$, $d_D(v) = |V(D)| - 3$. As in Case 1, we can prove that D^v is unicyclic. If G has a nontrivial component other than D, then G^v is not unicyclic, which is a contradiction. This implies that $G = D \cup (p - |V(D)|)K_1$ and Statement (5) is proved.

When $d_G(v) = |V(D)| - 2$, $d_D(v) = |V(D)| - 2$. Using Theorem 1.7, D^v is a tree. Since G is disconnected, G^v has $k(G)$ branches at v and the branch D^v is a tree, where $k(G)$ is the number of components of G. Since

G^v is connected and unicyclic, precisely one branch, B_1, at v in G^v is unicyclic. Let D_1 be the component of G such that $B_1 = D_1 + v$. Then, $D_1 = K_2$ since otherwise G^v is not unicyclic. Furthermore, the other components of G are trivial graphs. This implies that $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, proving Statement (6).

Conversely, let either (1), (2), (3), (4), (5), or (6) hold. From (1)–(6), we see that either $d_G(v) = 0$ or $d_B(v) < |V(B)| - 1$ for the unique branch B at v in G. From Theorem 1.6, G^v is connected. It is noted that for the branch B at v in G, if $d_B(v) = |V(B)| - 2$, then B^v is a tree or a unicyclic branch at v in G^v. Furthermore, if $d_B(v) = |V(B)| - 3$ and $G - v$ is acyclic, then B^v is a unicyclic branch at v in G^v. Each case implies that G^v is unicyclic. ■

Theorem 2.2: Let v be a cutvertex of a graph G of order $p \geq 3$. Let $D \neq K_1, K_2$ be a component of G that contains v. Then G^v is connected and unicyclic if and only if any one of the following statements is true:

(1) G is connected, $G - v$ is acyclic, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for any branch B at v in G, and $d_B(v) = |V(B)| - 3$ for only one B.

(2) G is connected, $G - v$ is unicyclic, $d_B(v) = |V(B)| - 2$ for any branch B at v in G, and $B - v$ is unicyclic for one B.

(3) $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, $G - v$ is acyclic, and $d_B(v) = |V(B)| - 2$ for any branch B at v in D.

(4) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is acyclic, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for any branch B at v in D, and $d_B(v) = |V(B)| - 3$ for only one B.

(5) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for any branch B at v in D, and $B - v$ is unicyclic for one B.

Proof: Let G^v be connected and unicyclic. From Theorem 1.6, $d_B(v) \leq |V(B)| - 2$ for every branch B at v. Since G^v is unicyclic, $G - v$ is either unicyclic or acyclic. G may be either connected or disconnected.

Case 1: G is connected.

Let k be the number of branches at v in G. By Corollary 1.4, there are only k branches at v in G^v since G^v is connected. Since G^v is unicyclic, exactly one branch, B_1, at v in G^v is unicyclic and all other branches at v in G^v are trees. Let $B^* = B_1^v$. Then B^* is a branch at v in G. By Theorem 1.7, $d_B(v) = |V(B)| - 2$ for every branch $B \neq B^*$ at v in G. If $d_{B^*}(v) < |V(B^*)| - 3$, then G^v is not unicyclic (see the proof in Case 1 of Theorem 2.1). This implies that $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$. If $G - v$ is acyclic, then $d_{B^*}(v) = |V(B^*)| - 3$, and if $G - v$ is unicyclic, then $d_{B^*}(v) = |V(B^*)| - 2$ and $B^* - v$ is unicyclic. This proves Statements (1) and (2).

Case 2: G is disconnected and $G - v$ is acyclic.

Let D be a component of G containing v. Since v is a cut vertex, D is neither K_1 nor K_2. Let B_1 be the unicyclic branch at v in G^v. We consider the two subcases, $B_1 = K_3$ and $B_1 \neq K_3$.

When $B_1 = K_3$, by Lemma 1.5, $B_1 - v = K_2$ is a component of G. Since G^v is unicyclic, each component, other than D and K_2, is K_1. This implies that $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$. Let B be any branch at v in G. Since G^v is connected, B^v is a branch at v in G^v. The unicyclic branch at v in G^v corresponds to the component K_2 of G, and hence, the branch B^v is a tree at v in G^v. By Theorem 1.7, $d_B(v) = |V(B)| - 2$, proving Statement (3).

When $B_1 \neq K_3$, let F be a component of G not containing v such that $F + v = B_1$. Then, by Lemma 1.5, $|V(F)| > 2$, which implies that B_1 is not unicyclic. This is a contradiction to our assumption that B_1 is unicyclic. Hence, B_1 is obtained from a branch, B^*, at v in G and $B_1 = B^{*v}$. Now B^* is connected and $B^* - v$ is acyclic. Applying Theorem 2.1(2) to B^*, we obtain $d_{B^*}(v) = |V(B^*)| - 3$. Let $B \neq B^*$ be a branch at v in G. Clearly, $d_B(v) = |V(B)| - 2$ since otherwise the branch B^v at v in G^v has a cycle. Furthermore, each component of G other than D is a trivial graph. This implies that $G = D \cup (p - |V(D)|)K_1$, proving Statement (4).

Case 3: G is disconnected and $G - v$ is unicyclic.

Clearly, v is in a nontrivial component, D, of G, since otherwise G^v is not unicyclic. Furthermore, the other components of G are trivial graphs. This implies that $G = D \cup (p - |V(D)|)K_1$, and hence, $G^v = D^v \cup (p - |V(D)|)(K_1 + v)$. Since G^v is unicyclic, D^v is unicyclic. Since D is connected and $D - v$ is unicyclic, by applying Case 1 to D, we obtain $d_B(v) = |V(B)| - 2$ for every branch B at v in D and $B - v$ is unicyclic for one B. Thus, Statement (5) is proved.

Conversely, let either (1), (2), (3), (4), or (5) hold. From (1)–(5), we see that $d_B(v) \le |V(B)| - 2$ for every branch B at v in G. Hence, by Theorem 1.6, G^v is connected. Because each case implies that G^v is unicyclic, the theorem is proved. ■

At this point we introduce notation from **[3]** to help describe the graphs we analyze in the sequel. Let $C_{r(v_1)}$ denote a cycle graph of length r oriented clockwise starting at v_1. We use the ordered list $C_{r(v_1)}(B_1, B_2, \ldots, B_r)$ to represent the graph formed by adding a branch B_i to vertex v_i in $C_{r(v_1)}$. For example, the graphs $C_{4(v)}(0, 0, P_2, P_3)$, $C_{4(v)}(0, 2P_2 \cup P_3, 0, 0)$, and $C_{4(v)}(0, 2P_2 \cup P_3, P_2, P_3)$ are illustrated in Figure 1.

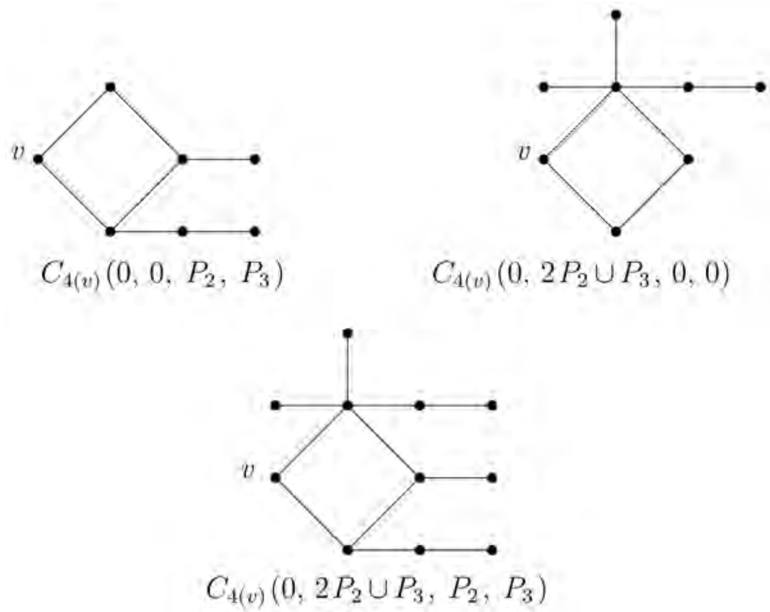

Figure 1

Now let v be a cut vertex of a connected graph G and let $B_1, B_2, \ldots, B_k$ be branches with $n_1, n_2, \ldots, n_k$ copies at v in G, respectively. In this case, we denote the graph G by $G(v; n_1B_1, n_2B_2, \ldots, n_kB_k)$. For example, in the graph G shown in Figure 2 there are four distinct branches B_1, B_2, B_3, and B_4 at v in G. These branches are shown individually in Figure 3. Thus, $G = G(v; 2B_1, B_2, B_3, B_4)$.

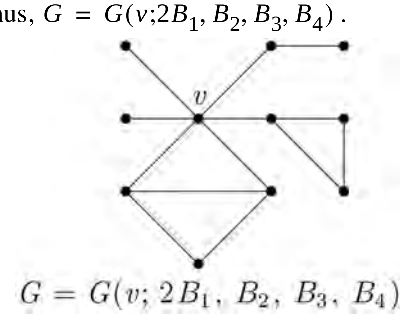

Figure 2

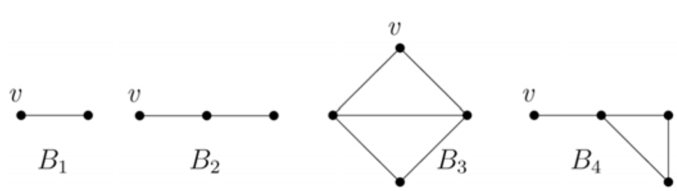

Figure 3

Theorem 2.5: Let G be a connected unicyclic graph of odd order $p = 2n + 1$. Then G has a self vertex switching v if and only if G is either of the following: $G(v;(n-2)P_3, C_{4(w)}(P_2, 0, 0, 0))$, $G(v;(n-2)P_3, C_{3(w)}(P_3, 0, 0))$, $G(v;(n-3)P_3, C_4, K_{1,3})$, or $G(v;(n-3)P_3, C_{3(w)}(P_2, 0, 0), P_4)$, where w is a vertex adjacent to v in G and, for every branch B at v in G, $d_B(v) = 1$ or 2 depending on whether B is a tree or unicyclic branch.

Proof: Let v be a self vertex switching of a connected unicyclic graph G. Clearly, $G \cong G^v$ and G has at least five vertices. Here, v may be either a cut vertex or not. We consider the two cases.

<u>Case 1</u>: v is not a cutvertex of G. By Theorem 2.1, either $G - v$ is acyclic and $d_G(v) = |V(G)| - 3$ or $G - v$ is unicyclic and $d_G(v) = |V(G)| - 2$.

When $G - v$ is acyclic and $d_G(v) = |V(G)| - 3$, v lies on the unicycle. Moreover, $d_G(v) = 2$ since G is connected and v is not a cutvertex of G byTheorem 1.1, with $p = 5$. There are only five unicyclic graphs on five vertices and these are Shown in Figure 4. It can be seen that $C_{4(w)}(P_2, 0, 0, 0)$ and $C_{3(w)}(P_3, 0, 0)$ are the only graphs with (two) self vertex switchings u and v.

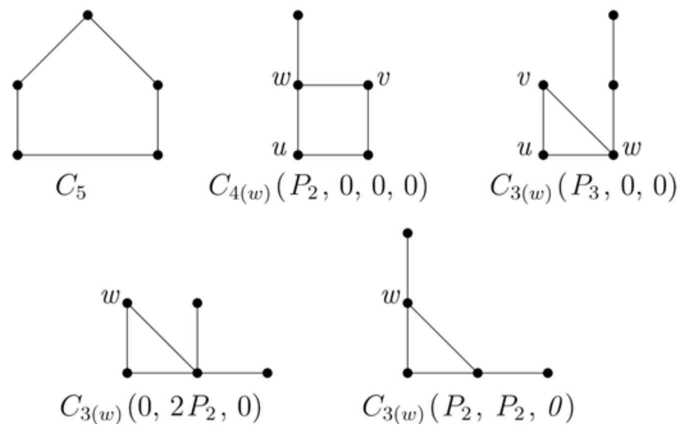

Figure 4

When $G - v$ is unicyclic and $d_G(v) = |V(G)| - 2$, v does not lie on the cycle of G because G is also unicyclic. Since $G - v$ is connected, v is an end vertex of G, otherwise G is not unicyclic and hence $p = 3$. The only unicyclic graph on three vertices is K_3, which does not have a vertex of degree 1. Hence, there is no connected unicyclic graph G such that $G - v$ is unicyclic and $d_G(v) = |V(G)| - 2$.

<u>Case 2</u>: v is a cutvertex of G. By Theorem 2.2, either $G - v$ is acyclic, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for every branch B at v in G, and $d_B(v) = |V(B)| - 3$ for only one B or $G - v$ is unicyclic, $d_B(v) = |V(B)| - 2$ for every branch B at v in G, and $B - v$ is unicyclic for one B. Thus, we consider two subcases.

<u>Case 2a</u>: $G - v$ is acyclic, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for every branch B at v in G, and $d_B(v) = |V(B)| - 3$ for only one B. In this case, v lies on the cycle in the cyclic branch, B^*, at v in G and $d_{B^*}(v) = 2$. By Theorem 1.1, $d_G(v) = n$. This implies that there are only $n - 1$ branches at v in G. Since G is unicyclic, any branch $B \neq B^*$ at v in G is a tree. Thus, from Theorem 1.2, $|V(B)| \geq 3$. If B^* has at least six vertices, then $p \geq (n-2)3 + 6 - (n-2) = 2n + 2 = p + 1 > p$, a contradiction. If B^* has three vertices, then $B^* - v$ is a component of G^v, and hence, G is disconnected. This implies that $|V(B^*)| = 4$ or 5.

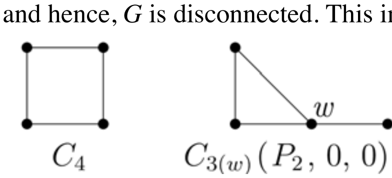

Figure 5

When $|V(B^*)| = 4$, B^* is either C_4 or $C_{3(w)}(P_2, 0, 0)$, since there are only two unicyclic graphs on four vertices. These are shown in Figure 5. For any vertex v with degree 2 in C_4 or in $C_{3(w)}(P_2, 0, 0)$, their switchings by v are $K_{1,3}$ or P_4, respectively, and in these the degree of v is 1. Clearly, C_4 and $K_{1,3}$; $C_{3(w)}(P_2, 0, 0)$ and P_4 are complementary switching branches at v. If B is a branch at v in G such that

$B \neq B^*$, B is not isomorphic with B^{*v}, and B has order at least 4, $p = |V(B^*)| + |V(B^{*v})| + |V(B)|$ plus the number of vertices in the remaining $n-4$ branches at v in G) minus $n+2$. This sum is at least $4+4+4+4+3(n-4)-n+2 = 2n+2 > p$, which is a contradiction. Hence, B has order 3. Since B is a tree, $B = P_3$. This implies that G is either $G(v;(n-3)P_3, C_4, K_{1,3})$ or $G(v;(n-3)P_3, C_{3(w)}(P_2, 0, 0), P_4)$.

When $|V(B^*)| = 5$, there are only five unicyclic graphs on five vertices, as shown in Figure 4. If B^* is one of C_5, $C_{3(w)}(0, 2P_2, 0)$, or $C_{3(w)}(P_2, P_2, 0)$, then for any v in B^* with degree 2, B^{*v} is unicyclic, and B^{*v} is not isomorphic with B^*. Hence, B^* is either $C_{4(w)}(P_2, 0, 0, 0)$ or $C_{3(w)}(P_3, 0, 0)$. By Case 1, both have a self vertex switching in the cycle and v is adjacent to w. Clearly, the other branches are P_3s. This implies that G is either $G(v;(n-2)P_3, C_{4(w)}(P_2, 0, 0, 0))$ or $G(v;(n-2)P_3, C_{3(w)}(P_3, 0, 0))$.

Case 2b: $G - v$ is unicyclic, $d_B(v) = |V(B)| - 2$ for every branch B at v in G, and $B - v$ is unicyclic for one B. $B - v$ is unicyclic and $d_B(v) = |V(B)| - 2$ implies that the branch B at v in G is not unicyclic. Hence, G is not unicyclic. This shows that there is no connected unicyclic graph G such that $G - v$ is unicyclic, $d_B(v) = |V(B)| - 2$ for every branch B at v in G, and $B - v$ is unicyclic for one B.

Consequently, G is one of the graphs $G(v;(n-2)P_3, C_{4(w)}(P_2, 0, 0, 0))$, $G(v;(n-2)P_3, C_{3(w)}(P_3, 0, 0))$, $G(v;(n-3)P_3, C_4, K_{1,3})$, or $G(v;(n-3)P_3, C_{3(w)}(P_2, 0, 0), P_4)$.

Conversely, let G be any of the graphs in the statement of the theorem. Then, clearly v is a self vertex switching of G. This completes the proof. ■

Corollary 2.6: Let G be a connected unicyclic graph. Then, $ss_1(G) = 0$ or 1 or 2. Moreover, $ss_1(G) = 1$ if and only if G is either $G(v;(n-2)P_3, C_{4(w)}(P_2, 0, 0, 0))$ and $n \geq 3$; $G(v;(n-2)P_3, C_{3(w)}(P_3, 0, 0))$ and $n \geq 4$; $G(v;(n-3)P_3, C_{3(w)}(P_2, 0, 0), P_4)$ and $n \geq 3$; or $G(v;(n-3)P_3, C_4, K_{1,3})$ and $n \geq 3$. Furthermore, $ss_1(G) = 2$ if and only if G is either $C_{4(w)}(P_2, 0, 0, 0)$, $C_{3(w)}(P_3, 0, 0)$, or $C_{3(w)}(0, P_3, P_3)$, where w is a vertex adjacent to the self vertex switching v in G. ■

Example 1: The connected unicyclic graphs corresponding to $n = 5$ ($p = 11$) are shown in Figure 6. In each graph the self vertex switching is at vertex v.

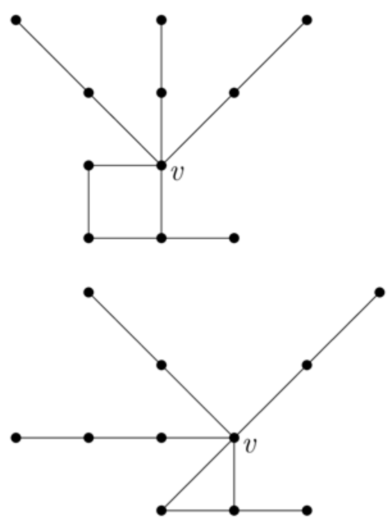

Figure 6

3. Characterizing Disconnected Unicyclic Graphs With a Self Vertex Switching

In this section we characterize vertices v such that G^v is disconnected, unicyclic, and has a specified number of components. We also characterize disconnected unicyclic graphs that have a self vertex switching.

Theorem 3.1: Let G be a graph of order $p \geq 3$. Let $D \neq K_1, K_2$ be a component of G containing v. Then G^v is disconnected and unicyclic with k components if and only if G has r branches at v, $d_B(v) = |V(B)| - 1$ for only $k-1$ branches at v, $r \geq k-1$, and one of the following statements is true:

(1) G is connected, $G - v$ is unicyclic, $r = k - 1$, and one $B - v$ is unicyclic.

(2) G is connected, $G - v$ is acyclic, $r > k - 1$, $d_B(v) \in \{|V(B)| - 2, |V(B)| - 3\}$ for the remaining $r - k + 1$ branches B at v in G, and $d_B(v) = |V(B)| - 3$ for only one B.

(3) G is connected, $G - v$ is unicyclic, $r > k - 1$, one $B - v$ is unicyclic, and $d_B(v) = |V(B)| - 2$ for the remaining $r - k + 1$ branches B at v in G.

(4) $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, $G - v$ is acyclic, and $r = k - 1$.

(5) $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, $G - v$ is acyclic, $r > k - 1$, and $d_B(v) = |V(B)| - 2$ for the remaining $r - k + 1$ branches B at v in G.

(6) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is acyclic, $r > k - 1$, $d_B(v) = |V(B)| - 3$ for only one branch B at v in G, and $d_B(v) = |V(B)| - 2$ for the remaining $r - k$ branches at v.

(7) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, $r = k - 1$, and one $B - v$ is unicyclic.

(8) $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, one $B - v$ is unicyclic, and $d_B(v) = |V(B)| - 2$ for the remaining $r - k + 1$ branches at v in G.

Proof: Let G^v be a disconnected and unicyclic graph with k components. By Theorem 1.8, G has $r \geq k - 1$ branches at v and $d_B(v) = |V(B)| - 1$ for $k - 1$ of these branches. Let $B_1, B_2, \ldots, B_{k-1}$ be the branches at v in G with $d_{B_i}(v) = |V(B_i)| - 1$, $1 \leq i \leq k - 1$. This implies that for any branch $B \neq B_i$, with $1 \leq i \leq k - 1$, $d_B(v) = |V(B)| - 2$. Since G^v is unicyclic, $G - v$ is acyclic or unicyclic. The graph G may be connected or disconnected. If G is connected, then $G - v$ is acyclic and $r = k - 1$. Hence, $G^v = K_1 \cup (\bigcup_{i=1}^{k-1} B_i)$ and v is K_1. This implies that G^v is not unicyclic since the graphs ($B_i - v$) are acyclic. Hence, there are seven cases to consider.

Case 1: G is connected, $G - v$ is unicyclic, and $r = k - 1$. In this case, $G^v = K_1 \cup (\bigcup_{i=1}^{k-1} B_i)$ and v is K_1. Since G^v is unicyclic, one $B_i - v$ is unicyclic and Statement (1) is true.

Case 2: G is connected, $G - v$ is acyclic, and $r > k - 1$.

Let G^* be the graph obtained from G by deleting the branches $B_1, B_2, \ldots, B_{k-1}$ except v. Then $G = G^* \cup (\bigcup_{i=1}^{k-1} B_i)$. From Theorem 1.3, $G^v = G^{*v} \cup (\bigcup_{i=1}^{k-1} B_i - v)$, since B_i^v is the union of the vertex v and $B_i - v$. Because G^v is unicyclic with k components and $G - v$ is acyclic, G^{*v} is unicyclic and connected. Let B_x be the unicyclic branch at v in G^{*v} and let B^* be the branch at v in G corresponding to the branch B_x at v in G^v. Applying Theorem 2.1(2) to B^*, $d_{B^*}(v) = |V(B^*)| - 3$. Furthermore, for any branch $B \neq B_i$, $1 \leq i \leq k - 1$ and $B \neq B^*$, $d_B(v) = |V(B)| - 2$, which proves Statement (2).

Case 3: G is connected, $G - v$ is unicyclic, and $r > k - 1$. Since $G - v$ is unicyclic, one $B - v$ is unicyclic. Let B^* be the branch at v in G such that $B^* - v$ is unicyclic. There are two subcases with respect to B^*.

Case 3a: $B^* = B_i$ for at least one i, $1 \leq i \leq k - 1$. As in Case 2, $G^v = G^{*v} \cup (\bigcup_{i=1}^{k-1} B_i - v)$. This implies that G^{*v} is a tree. By Theorem 1.7, $d_B(v) = |V(B)| - 2$ for any branch $B \neq B_i$ at v in G, $1 \leq i \leq k - 1$.

Case 3b: $B^* \neq B_i$, $1 \leq i \leq k - 1$. In this case, $d_{B^*}(v) \neq |V(B^*)| - 1$. Suppose $d_{B^*}(v) < |V(B^*)| - 2$. Then, G^v has at least two cycles; one contains v and the other does not. This contradicts our assumption that G^v is unicyclic. Hence, $d_{B^*}(v) = |V(B^*)| - 2$. Furthermore, $d_B(v) = |V(B)| - 2$ for $B \neq B_i$, $1 \leq i \leq k - 1$. Hence, Statement (3) is true.

To prove Statements (4)–(8), we assume that G is a disconnected graph with m components. Let these components be $D, D_1, D_2, \ldots, D_{m-1}$ and let v be in D. Let $D^* = D_1 \cup D_2 \cup \ldots \cup D_{m-1}$, so that $G = D \cup D^*$. Since G has r branches at v, D also has r branches at v.

Case 4: $G - v$ is acyclic and $r = k - 1$. In this case, $D = \bigcup_{i=1}^{k-1} B_i$, so that $G = D^* \cup (\bigcup_{i=1}^{k-1} B_i)$. Hence, $G^v = (D^* + v) \cup (\bigcup_{i=1}^{k-1} B_i - v)$. Since G^v is unicyclic and each ($B_i - v$) is acyclic, then $D^* + v$ is unicyclic. Let $D_j \neq D$ be a nontrivial component of G for at least one j, $1 \leq j \leq m - 1$. Then, $D_j = K_2$, since otherwise G^v is not unicyclic. Moreover, the remaining components are trivial graphs. Thus, $G = D \cup K_2 \cup (p - 2 - |V(D)|)K_1$, which proves Statement (4).

Case 5: $G - v$ is acyclic and $r > k - 1$. In this case $G = D \cup D^*$, and thus, $G^v = D^v \cup (D^* + v)$. Now, D^v may be either acyclic or unicyclic.

If D^v is acyclic, one component of G is K_2 and each of the others is K_1, because G^v is unicyclic. This implies that $G = D \cup K_2 \cup (p-2-|V(D)|)K_1$. Furthermore, for any branch $B \neq B_i$, $1 \le i \le k-1$, $d_B(v) = |V(B)|-2$. This proves Statement (5).

If D^v is unicyclic, each component of G except D is a trivial graph. Implying that $G = D \cup (p-|V(D)|)K_1$. Now D is connected, D^v is unicyclic, and $D-v$ is acyclic. Applying Case 2 to D, then $d_B(v) = |V(B)|-3$ for only one B and for each of the other $r-k$ branches at v in G, $d_B(v) = |V(B)|-2$. This proves Statement (6).

Case 6: $G-v$ is unicyclic and $r = k-1$. Suppose a component of G not containing v is unicyclic. Then, G^v has more than two cycles, which is a contradiction. Hence, all cycles are in the component D. The other components of G are trivial graphs. This implies that $G = D \cup (p-|V(D)|)K_1$. Since G^v is unicyclic then $B_i - v$ is unicyclic for some $i \in \{1, \ldots, k-1\}$. Hence, Statement (7) is proved.

Case 7: $G-v$ is unicyclic and $r > k-1$. Clearly, each component of G other than D is a trivial graph. This implies that $G = D \cup (p-|V(D)|)K_1$. Applying Case 3 to D, we observe that one $B-v$ is unicyclic and $d_B(v) = |V(B)|-2$ for the remaining $r-k+1$ branches at v in G. This proves Statement (8).

For the converse, using Theorem 1.8, G^v is disconnected with k components. Clearly, each case implies that G^v is unicyclic. ■

Theorem 3.2: A disconnected and unicyclic graph G of order $p = 2n+1$ with k components has a self vertex switching v if and only if it is either $D \cup K_2 \cup (k-2)K_1$, where D is either $D(v;(n-2)P_2, K_3)$ if $k = n$; or $D(v;(n-2)P_2, K_3, (n-k)P_3)$ if $k < n$; or $D \cup (k-1)K_1$ and $k = p+1-|V(D)|$, where D is one of the following:
$D(v;C_4, K_{1,3}, (k-1)P_2, (n-k-2)P_3)$,
$D(v;C_{3(w)}(P_2, 0, 0), P_4, (k-1)P_2, (n-k-2)P_3)$,
$D(v;C_{4(w)}(P_2, 0, 0, 0), (k-1)P_2, (n-k-1)P_3)$, or
$D(v;C_{3(w)}(P_3, 0, 0), (k-1)P_2, (n-k-1)P_3)$,
where w is a vertex adjacent to v in G and for any branch B at v in G, $d_B(v) = 1$ or 2, depending on whether B is a tree or unicyclic branch.

Proof: Let v be a self vertex switching of a disconnected unicyclic graph G. By Theorem 1.1, $d_G(v) = n$, and by Theorem 3.1, G has $r \ge k-1$ branches at v and $d_B(v) = |V(B)|-1$ for only $k-1$ branches B. There are five cases to consider.

Case 1: $G = D \cup K_2 \cup (p-2-|V(D)|)K_1$, $G-v$ is acyclic, and $r = k-1$, where $D \neq K_1, K_2$ is a component of G containing v. Since $G-v$ is acyclic, then v lies on the cycle in the cyclic branch, B^*, at v in G. Since G is unicyclic and $d_G(v) = n$, there are $n-1$ branches at v in G, and hence, $k = n$. Since K_2 is a component of G, $K_2 + v = K_3$ is a branch at v in G^v, and hence, $B^* = K_3$. Since the other branches at v in G are trees and $d_B(v) = |V(B)|-1$, each branch is P_2. These $n-2$ branches create only K_1 in G^v. Since $G \cong G^v$, $n-2 = p-2-|V(D)|$. Hence, $G = D \cup K_2 \cup (n-2)K_1$, where D is $D(v;(n-2)P_2, K_3)$.

Case 2: $G = D \cup K_2 \cup (p-2-|V(D)|)K_1$, $G-v$ is acyclic, $r > k-1$, and $d_B(v) = |V(B)|-2$ for the remaining $r-k+1$ branches at v in G, where $D \neq K_1, K_2$ is a component of G containing v. Let B^* be the unicyclic branch at v in G. As in Case 1, $B^* = K_3$ and there are $n-1$ branches at v in G. Clearly, $d_{B^*}(v) = |V(B^*)|-1$. Since the other branches at v in G are trees, any branch B at v in G with $d_B(v) = |V(B)|-2$ is P_3 and with $d_B(v) = |V(B)|-1$ is P_2. Since the P_2 branches at v in G generate only K_1s in G^v and $G \cong G^v$, then $k-2 = p-2-|V(D)|$. Hence, $G = D \cup K_2 \cup (k-2)K_1$, where D is $D(v;(k-2)P_2, K_3, (n-k)P_3)$. Furthermore, $n > k$ since $r = n-1$.

Case 3: $G = D \cup (p-|V(D)|)K_1$, $G-v$ is acyclic, $r > k-1$, $d_B(v) = |V(B)|-3$ for only one branch at v in G, and $d_B(v) = |V(B)|-2$ for the remaining $r-k$ branches at v in G, where $D \neq K_1, K_2$ is a component of G containing v. G is unicyclic and $G-v$ is acyclic implies that v lies on the cycle in the cyclic branch, B^*, at v in G. Hence, there are $n-1$ branches at v in G and $d_{B^*}(v) = 2$. If B^* has three vertices, then B^*-v is a component of G^v. Since $G \cong G^v$, G has a component K_2, which is a contradiction. Hence, $|V(B^*)| \ge 4$. By Lemma 1.5, each component $K_1 \neq D$ in G becomes a branch P_2 at v in G^v. If B is a branch at v in G with $d_B(v) \neq |V(B)|-1$, then B^v is a branch at v in G^v. By Lemma 1.5, $d_B(v) = |V(B)|-1$ for a branch B at v in G if and only if $B-v$ is a component of G^v. Since $G \cong G^v$ and there are $k-1$ branches at v in G with $d_B(v) = |V(B)|-1$, then $k-1 = p-|V(D)|$. If B is a branch at v in G with $d_B(v) \neq |V(B)|-1$, then $|V(B)| \ge 3$. If B^* has order at least 6, then

$$p = |V(B^*)| + \sum_{d_B(v) = |V(B)| - 1} |V(B)| + \sum_{d_B(v) \neq |V(B)| - 1,\, B \neq B^*} |V(B)| + x - r + 1,$$

where x is the number of isolated vertices in G. Hence,

$$p \geq 6 + 2(k-1) + 3(r-k) + p - |V(D)| - r + 1 = 2r + 4 = 2(n-1) + 4 = 2n + 2 > p,$$

which is a contradiction. Hence, $|V(B^*)| = 4$ or 5.

If $|V(B^*)| = 4$, there are only two unicyclic graphs on four vertices, namely, C_4 and $C_{3(w)}(P_2, 0, 0)$. For any vertex v with degree 2 in C_4 or in $C_{3(w)}(P_2, 0, 0)$, the switching by v is $K_{1,3}$ or P_4, respectively, and in each case the degree of v is 1. Clearly, C_4 and $K_{1,3}$, and $C_{3(w)}(P_2, 0, 0)$ and P_4 are complementary switching branches at v. Let B be any branch other than B^* that is not isomorphic with B^{*v}. If $d_B(v) = |V(B)| - 1$, then $B = P_2$, since otherwise $B - v \neq K_1$ is a component of G^v, and hence, G has a nontrivial component other than D. If $d_B(v) = |V(B)| - 2$, then B has three vertices, since otherwise, G has more than p vertices and hence, $B = P_3$. This implies that $G = D \cup (k-1)K_1$, where D is either

$$D(v; C_4, K_{1,3}, (k-1)P_2, (n-k-2)P_3) \text{ or } D(v; C_{3(w)}(P_2, 0, 0), P_4, (k-1)P_2, (n-k-2)P_3),$$

where vertex w is adjacent to v in G.

If $|V(B^*)| = 5$, there are only five unicyclic graphs on five vertices. These are shown in Figure 4. If B^* is either $C_{3(w)}(P_2, P_2, 0)$, $C_{3(w)}(0, 2P_2, 0)$, or C_5, then for any v in B^* with degree 2, B^{*v} is unicyclic and is not isomorphic with B^*. Hence, B^* is either $C_{4(w)}(P_2, 0, 0, 0)$ or $C_{3(w)}(P_3, 0, 0)$, has a self vertex switching in the cycle that is adjacent to w, and $d_{B^*}(v) = |V(B^*)| - 3$. Consider any branch $B \neq B^*$. If $d_B(v) = |V(B)| - 1$, then $B = P_2$, or if $d_B(v) = |V(B)| - 2$, then $B = P_3$, since otherwise G has more than p vertices. This implies that $G = D \cup (k-1)K_1$, where D is either

$$D(v; C_{4(w)}(P_2, 0, 0, 0), (k-1)P_2, (n-k-1)P_3) \text{ or } D(v; C_{3(w)}(P_3, 0, 0), (k-1)P_2, (n-k-1)P_3),$$

where vertex w is adjacent to v in G.

Case 4: $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, $r = k - 1$, and one $B - v$ is unicyclic, where $D \neq K_1, K_2$ is a component of G containing v. Let B^* be the branch at v in G such that $B^* - v$ is unicyclic. Since $d_{B^*}(v) = |V(B^*)| - 1$, $B^* - v$ is a component of G^v. Since $G \cong G^v$, then $D \cong B^* - v$, which contradicts the fact that $B^* - v$ is a proper subgraph of D. In this case, G cannot have a self vertex switching.

Case 5: $G = D \cup (p - |V(D)|)K_1$, $G - v$ is unicyclic, $r > k - 1$, one $B - v$ is unicyclic, and $d_B(v) = |V(B)| - 2$ for the remaining $r - k + 1$ branches at v in G. $D \neq K_1, K_2$ is a component of G containing v. Because G and $G - v$ are unicyclic, then v does not lie on the cycle in the unicyclic branch, B^*, at v in G. Hence, v is an end vertex in B^*. Consequently, $|V(B^*)| > 3$. Since $d_{B^*}(v) = |V(B^*)| - 1$ or $|V(B^*)| - 2$, $d_{B^*}(v) > 1$, which contradicts the fact that v is an end vertex in B^*. In this case also, G has no self vertex switching.

Cases 1–3 generate the graphs in the statements of the theorem.

Conversely, if G is any of the graphs in the statement of the theorem, then v is the self vertex switching of G. This completes the proof. ■

Corollary 3.3: Let G be a disconnected unicyclic graph. Then $ss_1(G) = 0$ or 1. Furthermore, $ss_1(G) = 1$ if and only if G is one of the graphs given in Theorem 3.2. ■

Example 2: The five disconnected unicyclic graphs on 17 vertices, each of which has v as the self vertex switching and has three components, are shown in Figure 7.

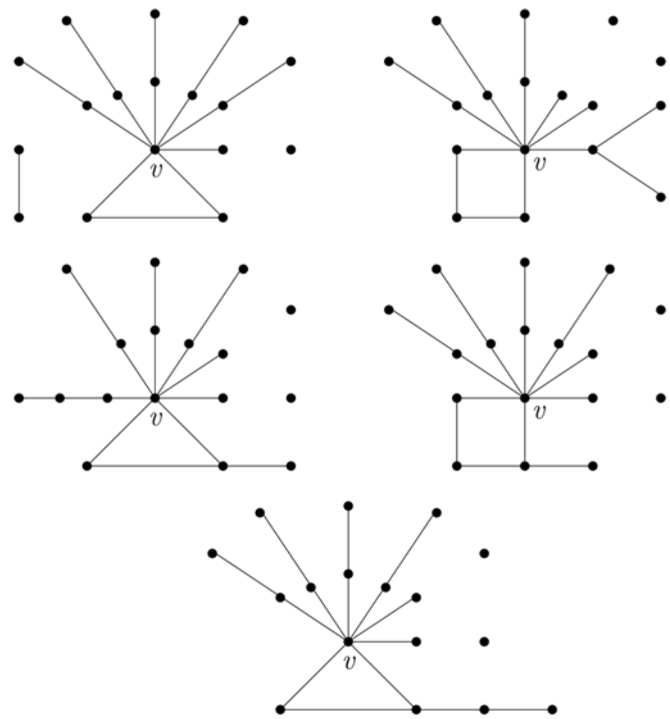

Figure 7

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[GTN LXII:5] CHARACTERIZING THE STRUCTURE OF GRAPHS WITH EQUAL DOMINATION PARAMETERS

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Abstract

In this paper we provide a direct proof for the characterization of the structure of graphs with upper domination number equal to the uniform domination number.

1. Introduction

All graphs considered in this paper are finite, undirected, simple graphs. We follow the standard definitions of graph theory as found in **[1]**.

Let $G = (V, E)$ be a graph. A subset S of V is called a *dominating set* if every vertex is either in S or is adjacent to at least one vertex in S. A dominating set S is *minimal* if no subset of S is a dominating set. The *domination number* γ and *upper domination number* Γ of a graph G are, respectively, the minimum and maximum cardinalities taken over all minimal dominating sets of G. The *uniform domination number* γ_u of G is the least positive integer k such that every k-element subset of V is a dominating set of G. The concept of uniform domination in graphs was introduced by Arumugam and Joseph **[2]**.

The *open neighborhood* $N(v)$ of a vertex v consists of the set of all vertices adjacent to v and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. We denote the minimum and maximum degree of a graph G by $\delta(G)$ and $\Delta(G)$, respectively.

A subset S of V in a graph G is called an *independent set* if no two vertices in S are adjacent. The maximum cardinality of an independent set of vertices of G is called the *vertex independence number* of G and is denoted by $\beta_0(G)$.

Let S be a set of vertices of a graph G and let $u \in S$. We say that a vertex v is a *private neighbor* of u (with respect to S) provided $N[v] \cap S = \{u\}$. A subset S of V is called *irredundant* if every vertex $u \in S$ has at least one private neighbor. An irredundant set S is called a *maximal irredundant set* if no superset of S is irredundant. The maximum cardinality over all maximal irredundant sets of vertices of G is called the *upper irredundance number* of G and is denoted by $IR(G)$.

The *join* of n vertex disjoint graphs $G_1, G_2, \ldots, G_n$ with vertex sets $V_1, V_2, \ldots, V_n$, respectively, is denoted by $G_1 + G_2 + \ldots + G_n$ and is defined to be $G_1 \cup G_2 \cup \ldots \cup G_n$ together with all edges connecting V_i and V_j for $1 \le i < j \le n$.

A graph G is said to be *decomposable* if G can be expressed as the join of two proper subgraphs (equivalently, if $\overline{G}$, the complement of G, is disconnected). A graph that is not decomposable is said to be *indecomposable*. If G can be decomposed into r proper vertex disjoint subgraphs $G_1, G_2, \ldots, G_r$ such that each $G_i \cong H$, then G can be written as $H^{(r)}$. In this paper, we use $H_r = \overline{K_{r/2,\, r/2} - M}$, where $r \ge 4$ and M is a perfect matching in $K_{r/2,\, r/2}$.

For any set S of vertices of G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S.

We need the following results.

Theorem 1.1 [2]: Let G be a graph of order p, then $\gamma_u = p - \delta$. ■

Theorem 1.2 [3]: Let G be a graph of order p and minimum degree $\delta(G)$, $IR(G) \le p - \delta(G)$, where equality holds if and only if G is one of the following graphs:

(1) $V(G) = X \cup W$, where X is independent in G and each vertex in X is joined to each vertex in W. That is, $G = X + W$. The vertices in W are arbitrarily adjacent to each other subject to the restriction that $\deg(w) \ge \delta$ for each $w \in W$.

(2) $V(G) = X \cup Y \cup Z$, where $|X| = |Y| = p - \delta$ (that is, $\delta \ge p/2$), $\langle X \rangle \cong \langle Y \rangle \cong K_{p-\delta}$, and the vertices in X are joined to the vertices in Y by a matching. Furthermore, $|Z| = 2\delta - p$ and each vertex in Z is joined to each vertex in $X \cup Y$. The vertices in Z are arbitrarily adjacent to each other subject to the restriction that $\deg(z) \ge \delta$ for each $z \in Z$. ■

Theorem 1.3 [4]:
(1) If $IR(G) = p - \delta(G)$, then $\Gamma(G) = IR(G)$.

(2) If $IR(G) = p - \delta(G)$, where $\delta(G) < p/2$, then $\beta_0(G) = \Gamma(G) = IR(G)$. ■

2. Main Results

We characterize the structure of graphs for which the upper domination number is equal to the uniform domination number. That is, graphs G such that $\gamma_u(G) = \Gamma(G)$.

The following results are obvious.

Lemma 2.1: $\gamma_u(P_m + \overline{K_n}) = \Gamma(P_m + \overline{K_n})$ if and only if $1 \le m \le n + 1$. ■

Lemma 2.2: $\gamma_u(C_m + \overline{K_n}) = \Gamma(C_m + \overline{K_n})$ if and only if $3 \le m \le n + 2$. ■

Lemma 2.3: $\gamma_u(K_2^{(m)} + \overline{K_n}) = \Gamma(K_2^{(m)} + \overline{K_n})$ if and only if $4 \le 2m \le n + 1$. ■

Theorem 2.4: Let G_1 and G_2 be two vertex disjoint graphs.
Then $\Gamma(G_1 + G_2) = \max\{\Gamma(G_1), \Gamma(G_2)\}$.

Proof: Let G_1 and G_2 be graphs with orders p_1 and p_2, and with minimum degrees δ_1 and δ_2, respectively. Let $G = G_1 + G_2$. Then, $\delta(G) = \min\{p_1 + \delta_2, p_2 + \delta_1\}$. Without loss of generality, let $\delta = p_1 + \delta_2$. As is customary, we refer to any minimal dominating set of maximum cardinality of a graph G as a Γ-set of G. Clearly, any Γ-set of G_1 or G_2 is also minimal dominating set of G. Consequently,

$$\Gamma(G) \ge \max\{\Gamma(G_1), \Gamma(G_2)\}.$$

Furthermore, if $\Gamma(G) \ge 2$, any Γ-set of G is also a Γ-set of either G_1 or G_2. Thus, if $\Gamma(G) \ge 2$, $\Gamma(G) \le \max\{\Gamma(G_1), \Gamma(G_2)\}$. If either $\Gamma(G) = 1$ or $\max\{\Gamma(G_1), \Gamma(G_2)\} = 1$, then G_1,G_2, and G are complete graphs, so that $\Gamma(G) = \max\{\Gamma(G_1), \Gamma(G_2)\} = 1$. Thus, $\Gamma(G) = \max\{\Gamma(G_1), \Gamma(G_2)\}$. ■

Next, we characterize the structure of indecomposable graphs for which $\gamma_u(G) = \Gamma(G)$.

Theorem 2.5: Let G be an indecomposable graph for which $\gamma_u(G) = \Gamma(G) = p - \delta(G)$.
Then, G is isomorphic with either $\overline{K_p}$ or $H_p = \overline{K_{p/2,\,p/2} - M}$, where p is an even integer greater than 4 and M is a perfect matching in $K_{p/2,\,p/2}$.

Proof: Let $D = \{v_1, v_2, \ldots, v_s\}$ be a minimal dominating set of G with $s = p - \delta(G)$ and let $S = V - D = \{u_1, u_2, \ldots, u_r\}$ with $|S| = \delta = r$. If $|D| = 1$, then $\delta = p - 1$ and $G = K_1 = \overline{K_1}$. Thus, assume that $|D| > 1$. There are two cases to consider.

Case (1): Suppose there is an isolated vertex v in D. Then, since $N(v) \subseteq V - D$, it follows that $N(v) = V - D$ because $|V - D| = \delta$. Thus, v is adjacent to every vertex in $V - D$. From the characterization of minimal dominating sets **[5]**, every vertex u in D is isolated in D, and hence, $N(u) = V - D$ for all u in D. If now $V - D \ne \varnothing$, it follows that $G = \langle V - D \rangle + \langle D \rangle$, which contradicts the indecomposability of G. Hence, $V - D = \varnothing$. Therefore, $G \cong \overline{K_p}$.

Case (2): Suppose that D has no isolated vertex. Then, for each vertex v_i in D, there is a vertex u_i in $V-D$ such that $N(u_i) \cap D = \{v_i\}$, $1 \le i \le s$. From degree considerations, it follows that

(1) $$N(u_i) = \{v_i\} \cup \{V - D - \{u_i\}\}.$$

In particular, $\langle V-D \rangle = \{u_1, u_2, \ldots, u_r\}$ is a complete subgraph of G. The number of vertices in $V-D$ that are adjacent to v_i is at most $\delta - (p - \delta - 1) = 2\delta - (p-1)$. Also, the number of vertices in D adjacent to v_i is at least $\delta - [2\delta - (p-1)] = p - \delta - 1$. Thus, v_i must be adjacent to all the vertices in $D - \{v_i\}$. Therefore, $\langle D \rangle$ is a complete subgraph of G. Consequently, V is the disjoint union of D and $V-D$, $|D| = |V-D|$, $\langle D \rangle$ and $\langle V-D \rangle$ are complete subgraphs, and $\{u_1v_1, u_2v_2, \ldots, u_sv_s\} = \langle D, V-D \rangle$ forms a perfect matching of G. Thus, $G \cong H_p$. Let

$$T = V - \{v_1, v_2, \ldots, v_s, u_1, u_2, \ldots, u_s\} = \{u_{s+1}, u_{s+2}, \ldots, u_r\}.$$

Claim: Every vertex in T is adjacent to every vertex in D. Suppose vertex u_{s+1} is not adjacent to some vertex v_i, for example v_1. Let $D^1 = \{u_{s+1}, u_2, \ldots, u_s\}$. Then, since v_1 is not adjacent to $u_2, \ldots, u_s, u_{s+1}$, D^1 is not a dominating set. However, $|D^1| = p - \delta(G) = \gamma_u(G)$, and we have a contradiction. Thus, u_{s+1} is adjacent to v_1. Similarly, every vertex in T is adjacent to every vertex in D. Let G be the graph illustrated in Figure 1, where, for $i = 1, 2$, each K_s^i represents a complete subgraph on s vertices.

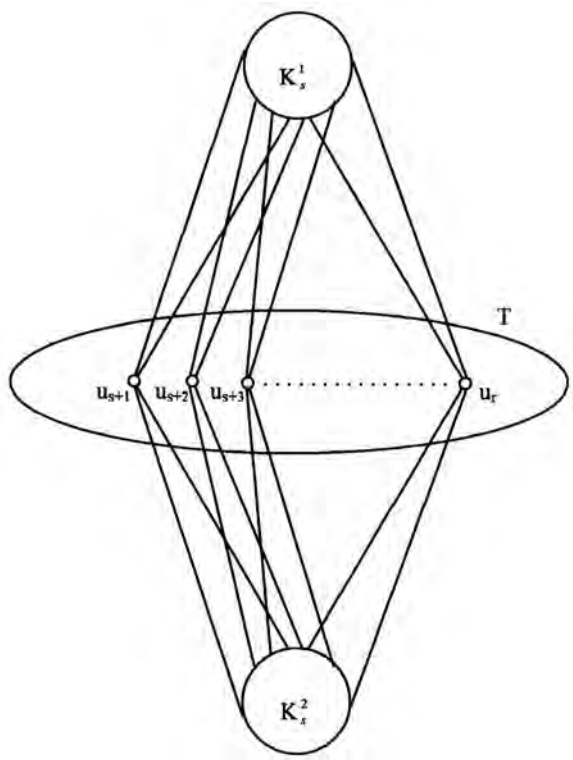


Figure 1

From Equation **(1)**, every vertex in T is adjacent to every u_i, $1 \le i \le s$. Therefore, $G = \langle T \rangle + \langle V-T \rangle$, which contradicts the indecomposability of G. Thus, $T = \varnothing$ and again $G \cong H_p$. However, if $p = 2$ or $p = 4$, then $G \cong K_2$ or $G \cong K_{2,2}$, respectively, which are decomposable graphs. If $p \ge 6$, these graphs H_p may be verified to be indecomposable. Thus, $G \cong H_p$, where p is an even integer greater than 4. ■

Next we characterize the structure of decomposable graphs for which $\gamma_u(G) = \Gamma(G)$.

Theorem 2.6: Let G be a decomposable graph with $\Gamma(G) = p - \delta(G)$. Then, there exist indecomposable graphs $G_1, G_2, \ldots, G_r$, where each $G_i \cong \overline{K_s}$ or $G_i \cong H_{2s}$, when $s = \delta(G) > 2$ such that $G = \overline{K_s}^{(m)} + H_{2s}^{(n)} + H$, where m and n are nonnegative integers and H is an arbitrary graph with $\Gamma(H) < \Gamma(G)$.

Proof: We argue by induction on p. If $p = 1$ or 2, then the theorem is true. Assume that the theorem is true for all decomposable graphs with $\Gamma(G) = p - \delta(G)$ and $p \le k$. Now, let G be a decomposable graph with $p = k + 1$. Let G_1 and G_2 be graphs with orders p_1 and p_2 and with minimum degrees δ_1 and δ_2, respectively. Let $G = G_1 + G_2$. We may assume that $p_1 - \delta_1 \ge p_2 - \delta_2$. There are two cases to consider.

Case (1): Without loss of generality, $p_1 - \delta_1 > p_2 - \delta_2$. Since $\delta(G) = \min\{p_1 + \delta_2, p_2 + \delta_1\}$, then $\delta(G) = p_2 + \delta_1$. Now, $\Gamma(G) = p - \delta(G) = (p_1 + p_2) - (p_2 + \delta_1) = p_1 - \delta_1$. We claim that $\Gamma(G_1) > \Gamma(G_2)$. Assume that $\Gamma(G_1) \le \Gamma(G_2)$. Then,

$$p_1 - \delta_1 = \Gamma(G) = \max\{\Gamma(G_1), \Gamma(G_2)\} = \Gamma(G_2) \le p_2 - \delta_2 < p_1 - \delta_1,$$

which contradicts the hypothesis for this case. Hence, $\Gamma(G_1) > \Gamma(G_2)$. This implies that $\Gamma(G_1) = p_1 - \delta_1$. If G_1 is indecomposable, then, by Theorem 2.5, $G_1 \cong \overline{K_s}$ or $G_i \cong H_{2s}$. Suppose that G_1 is decomposable. Since $\Gamma(G_1) = \Gamma(G) = p - \delta(G) = s$ and $|V(G_1)| = p_1 < k + 1$, then by induction G_1 can be decomposed in the form $G_1 = \overline{K_s}^{(b_1)} + H_{2s}^{(b_2)} + H_1$, where b_1 and b_2 are nonnegative integers and H_1 is an arbitrary graph with $\Gamma(H_1) < \Gamma(G_1)$. Thus, $G = \overline{K_s}^{(b_1)} + H_{2s}^{(b_2)} + H_2$, where b_1 and b_2 are nonnegative integers and $H_2 = H_1 + G_2$. We note that $\Gamma(H_2) < \Gamma(G_1) = \Gamma(G)$.

Case (2): $p_1 - \delta_1 = p_2 - \delta_2$. There are two cases to be considered.

Subcase 2.1: $\Gamma(G_1) \ne \Gamma(G_2)$. If $\Gamma(G_1) > \Gamma(G_2)$, then $\Gamma(G_1) = \Gamma(G) = s$ and $\Gamma(G_2) < \Gamma(G)$. Then, by an argument similar to that used in Case (1), the theorem is proved.

Suppose that $\Gamma(G_1) < \Gamma(G_2)$. Then, by interchanging G_1 and G_2 we may assume that $\Gamma(G_1) > \Gamma(G_2)$ and the theorem has been already proved.

Subcase 2.2: $\Gamma(G_1) = \Gamma(G_2)$. Since $\Gamma(G) = \Gamma(G_1) = \Gamma(G_2) = s = p_1 - \delta_1 = p_2 - \delta_2$, we can apply either Theorem 2.5 or induction to each of G_1 and G_2 (depending on decomposability) and prove the theorem.

Thus, in all cases G can be decomposed into indecomposable graphs $G_1, G_2, \ldots, G_r$, where each $G_i \cong \overline{K_s}$ or $G_i \cong H_{2s}$, such that $G = \overline{K_s}^{(c_1)} + H_{2s}^{(c_2)} + H$, where c_1 and c_2 are nonnegative integers and H is an arbitrary graph with $\Gamma(H) < \Gamma(G)$. Hence, by induction hypothesis, the theorem is true for any p. ■

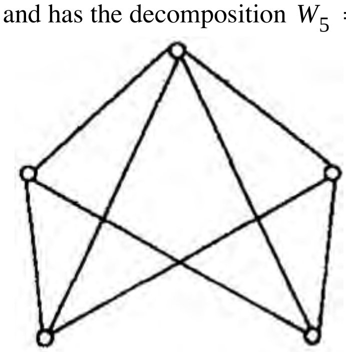
Remark 1:

(1) If $s = 1$, then G is complete and $G \cong K_1^{(p)}$.

(2) If $s = 2$, then $\Delta(G)$ is either $(p-1)$ or $(p-2)$. If $\Delta(G) = p - 1$, then $G \cong K_g + \overline{K_2}^{((p-g)/2)}$ for some g and if $\Delta(G) = p - 2$, then $G \cong \overline{K_2}^{(p/2)}$.

Remark 2: Theorems 2.5 and 2.6 combine to give the structure theorem for graphs G with $\Gamma(G) = p - \delta(G)$.

Remark 3: We illustrate the structural characterization with the following example. The wheel graph $W_5 = C_4 + K_1$ has $\Gamma = p - \delta = 2$ and has the decomposition $W_5 = \overline{K_2} + \overline{K_2} + K_1$, see Figure 2.



$\mathbf{W_5}$

Figure 2

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